TOPOLOGICAL OPTIMIZATION IN HYPERCYCLE BASED INTERCONNECTION NETWORKS

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Technical Report ECE 94-2

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April 1994

Abstract

A major step in designing a highly efficient and reliable interconnection network for parallel computer systems is finding a topology with minimum diameter and average distance for a given set of constraints such as the number of nodes, degree and dimension of the network. The average distance is related to the network latency and hence dictates the performance of a parallel computer. This work focuses on hypercycles which are a class of multidimensional, symmetric graphs for computer interconnection networks with variable diameter, simple routing and incremental expandability. These graphs encompass a wide spectrum of interconnection networks including the ring, torus, binary n-cube, k-ary n-cubes, generalized hypercubes and can be viewed as products of circulant graphs. In this report, we present new results on the synthesis of cost optimal hypercycle network topologies for average distance, diameter and product of degree and diameter. Analytical solutions are derived, wherever possible and these are compared with numerical results of the optimization.

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1. Introduction

The rapid development in VLSI technology in recent years has opened new avenues in the field of parallel computing. With decreasing hardware costs, it is now possible to interconnect several thousands of inexpensive processors for scientific and engineering applications [1]. A parallel computer consists of a collection of interconnected computing nodes that co-operate in the execution of a particular task. The design of parallel computers poses several challenging problems in network topology selection, communication hardware design, fault-tolerant aspects, operating systems and algorithm design.

A plethora of interconnection topologies, including *bus, ring, chordal ring, complete connection, mesh, torus, omega, lens, X-tree, hypercube, generalized hypercubes, and cube-connected cycles* have been proposed in the open literature [3]. An ideal network design should normally possess low node connectivity, small diameter, minimum average distance, regularity, modularity, expandability, ease of implementation, simple routing schemes, fault-tolerance and reconfiguration capability. The binary n-cube or hypercube has been proven to be efficient with significant speedups in many real time applications such as computational aerodynamics, quantum physics and image processing. Some examples of hypercube machines include the Intel's iPSc [4], NCUBE/10 [5], Caltech's Cosmic Cube [6], the Connection Machine [7] and MAX [8].

Embedded real time applications [8, 9] tend to exhibit varied structures that do not necessarily map optimally onto the hypercubes. The binary n-cube always expands as a power of 2 and the number of communication links at each node is given by n. This constitutes a significant increase in resource allocation especially in space, mass, power-limited environments (eg. spacecrafts and submersibles). Recently, a new class of multidimensional graphs called hypercycles [10] has been introduced for computer interconnection networks. Hypercycles are symmetric graphs that are generalizations of hypercubes and embody a cyclic interconnection topology with variable connectivity, incremental expandability and simple routing. These networks have the ability, given a fixed degree, to form a number of alternate size graphs and therefore can be tailored to match the topology of a given application.

Many real-time programs are I/O bound [2] and their execution on a parallel computer is often limited by the network's communication capability. The total execution time of an application in a massively parallel machine is composed of the computation time and communication delay or network latency as dictated by *Amdahl's law* [11]. In a massively parallel system, the communication overhead should be minimal and comparable to the computation time, and the system throughput should be sufficiently high to permit data sharing.

A major factor in realizing a highly efficient and reliable network is finding a graph with minimum diameter and average distance and maximal connectivity for a given set of nodes and degree [12]. The diameter and hence the average distance is related to the network latency while connectivity is related to the fault-tolerance of the system.

Technological constraints limit the number of I/O ports in each node of a parallel computer and this places an upper bound on the communication bandwidth. From the stand point of logical and physical realizability, it is advantageous to have minimum connectivity but at the cost of increasing the network latency. Therefore, a pragmatic and viable approach to design a network system with high performance and minimum cost, should consider both connectivity and diameter. This often necessitates the introduction of optimization techniques in network design [13]-[17].

The Generalized Hypercubes and Hyperbus structures (GHC) were introduced by Bhuyan *et. al* [17] where an analysis was made in realizing a cost-optimal network based on the degree and diameter of the GHC. Dally [2] considered the optimization of the network latency and throughput for k-ary n-cubes incorporating worm-hole routing. Since hypercycles represent a class of graphs rather than isolated graphs, they are more general than GHC encompassing a wide spectrum of topologies such as the torus, hypercubes, k-ary n-cubes and GHC to mention a few. Due to the increased variability and complexity, the design of hypercycles with optimal performance-to-cost ratio is more difficult.

In this report, we present an analysis that establishes the optimality of balanced hypercycles (those which have equal node population and connectivity in each dimension) with respect to the average distance, diameter and product of degree and diameter. In addition, it is shown that under certain conditions, the optimum corresponds to hypercycles which are not balanced. The work is organized as follows. Section 2 introduces the terminology used in this analysis along with a brief overview of hypercycles and their properties. Section 3 deals with the formulation of the average distance as an optimization problem in hypercycle networks. In Section 4, the optimization of the average distance and diameter is considered. Analytical solutions are presented and compared with numerical results. Section 5 considers special cases of the formulated optimization problem while the product of degree and diameter is analyzed in Section 6. Section 7 briefly discusses about network latency and throughput in hypercycles. Finally, the report concludes with a short discussion of the obtained results and their significance.

2. Definitions and Preliminaries

2.1. Terminology

The notations used extensively in this work are summarized for clarity:

Abbreviations

DRQP – Discrete Recursive Quadratic Programming
GHC – Generalized Hypercubes
MRN – Mixed Radix Number System

Design Parameters

r – Dimension of a hypercycle (integer)
m_i – Node radix of a hypercycle in dimension i (integer)
ρ_i – Connectivity in dimension i (integer)

Constants

M – Total number of nodes in a hypercycle

Design Functions

k – diameter of a hypercycle
k_b – diameter of balanced hypercycle
k' – diameter of an unbalanced hypercycle
d – degree of a hypercycle
δ – Average distance of a hypercycle
δ_b – Average distance of an balanced hypercycle
δ' – Average distance of an unbalanced hypercycle
χ – Product of hypercycle degree and diameter

Other Variables

S_j – Sum of distances of all nodes in the j^{th} dimension from reference node
D_l – Shortest distance of node l from reference node
$\hat{\delta}$ – Upper bound on the average distance of a hypercycle
$\check{\delta}$ – Lower bound on the average distance of a hypercycle
$\hat{r}$ – Upper bound on the dimension of a hypercycle
$\check{r}$ – Lower bound on the dimension of a hypercycle

Performance Metrics

T – Network Latency
T' – Normalized Network Latency
T_d – Traffic Density
C_p – Network Capacity
T_p – Network Throughput
T'_p – Normalized Network Throughput
σ – Communication overhead

2.2. Hypercycles

A simple, finite, undirected graph is represented by $G = (V,\ E)$ where $V(G)$ is the *node set* while $E(G)$ denotes the *edge set* [18]. The *Cartesian product* G of two graphs G_1 and G_2 is given by $G_1 \times G_2$ where $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent i.e. an edge connecting them exists if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$. The order of a graph is the number of nodes denoted by $|\mathcal{G}|$. Two graphs G and H are *isomorphic* if there is a bijection $\phi : V(G) \rightarrow V(H)$ such that $(x, y) \in E(G)$ iff $(\phi(x), \phi(y)) \in E(H)$.

The *connectivity* of G is defined as the minimum number of edges of G whose removal disconnects G. The *degree* d of a node v in a graph G is the number of edges of G incident to v. A graph G is *regular* if all nodes have the same degree. A graph G is *complete* if every pair of its nodes is adjacent. The distance between two nodes is the number of edges separating them. The *diameter* k is the largest value of the shortest distance separating any two nodes in G.

The Mixed Radix Number (MRN) system [14] is used in defining the topology of a hypercycle graph. Given a decimal number M which can be expressed as a product of r factors such that $M = \prod_{j=1}^{r} m_j$, then any number $0 \leq X \leq M-1$ can be expressed as a vector given by

$$\mathrm{MRN}(X)_{m_1\ m_2\ \ldots\ m_r} = (x_1\ x_2\ \ldots\ x_r),\quad 0 \leq x_i \leq m_i - 1,\quad i = 1, \ldots r$$

Each x_i is chosen such that

$$X = \sum_{i=1}^{r} x_i w_i \quad \text{where } w_i = \frac{M}{\prod_{j=1}^{i} m_j} = \prod_{j=i+1}^{r} m_j$$

An r-D hypercycle [20] is defined by $\mathcal{G}_{\vec{m}}^{\vec{\rho}} = \{\mathcal{V}_{\vec{m}}^{\vec{\rho}}, \mathcal{E}_{\vec{m}}^{\vec{\rho}}\}$ where

1. $\mathcal{V}_{\vec{m}}^{\vec{\rho}}$ is the set of nodes $\{0, 1, 2, \ldots M-1\}$ where $M = |\mathcal{G}_{\vec{m}}^{\vec{\rho}}|$.
2. $\mathcal{E}_{\vec{m}}^{\vec{\rho}}$ are the edges in the graph;
3. $\vec{m} = [m_1\ m_2\ \ldots\ m_r]$ is a basis vector such that $M = \prod_{i=1}^{r} m_i$ and $m_i \geq 2$ is an integer;
4. $\vec{\rho} = [\rho_1\ \rho_2\ \ldots\ \rho_r]$ is the connectivity vector where $1 \leq \rho_i \leq \lfloor \frac{m_i}{2} \rfloor$.

If $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ has vertices (α, β) such that

$$(\alpha)_{m_1 m_2 \ldots m_r} = \alpha_1\ \alpha_2 \ldots \alpha_i \ldots \alpha_r,\quad 0 \leq \alpha_i \leq m_i - 1$$
$$(\beta)_{m_1 m_2 \ldots m_r} = \beta_1\ \beta_2 \ldots \beta_i \ldots \beta_r,\quad 0 \leq \beta_i \leq m_i - 1$$

then $(\alpha, \beta) \in \mathcal{E}_{m}^{\vec{\rho}}$ iff there exists some j for $1 \leq j \leq r$ such that

$$\beta_j = (\alpha_j \pm \xi_j) \bmod m_j, \text{for } 1 \leq \xi_j \leq \rho_j$$
$$\alpha_i = \beta_i \quad \text{for } i \neq j$$

The degree d [10] of a hypercycle is given by

$$d = \sum_{i=1}^{r} d_i \tag{1}$$

where d_i is the degree in the i^{th} dimension such that

$$d_i = \begin{cases} 2\rho_i & \text{if } 2\rho_i < m_i \\ 2\rho_i - 1 & \text{if } 2\rho_i = m_i \end{cases}$$

The diameter k [20] of a hypercycle is given by

$$k = \sum_{i=1}^{r} k_i \tag{2}$$

where k_i is the diameter in the i^{th} dimension such that

$$k_i = \left\lceil \frac{\lfloor \frac{m_i}{2} \rfloor}{\rho_i} \right\rceil \tag{3}$$

The number of edges in $\mathcal{E}_{\vec{m}}^{\vec{\rho}}$ is given by

$$|\mathcal{E}_{\vec{m}}^{\vec{\rho}}| = \frac{Md}{2}$$

Hypercycles are essentially cartesian products of circulant graphs in each dimension and some examples are illustrated in Fig. 1.

Definition 1 *A hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is said to be* **balanced** *if $m_1 = m_2 = \ldots = m_r$ and $\rho_1 = \rho_2 = \ldots = \rho_r$.*

Definition 2 *The connectivity-sum (C) of a hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is defined as the sum of its connectivity in each dimension such that*

$$C \;=\; \sum_{i=1}^{r} \rho_i \geq \frac{d}{2}$$

Definition 3 *The solution space Ω of the node radix vector $\vec{m}$ for a given M is the set of all possible integer values of $m_1, m_2, \ldots, m_r$ such that $M = \prod_{i=1}^{r} m_i$.*

Definition 4 *The neighbourhood of a point, $\vec{m}^* = [m_1 \; m_2 \; \ldots \; m_r]$ in the solution space Ω of the node radix vector for a given $\vec{\epsilon}$ is a region $\mathcal{R}$ enclosing a set of points $\vec{m} \in \Omega$ defined by*

$$\mathcal{R}_{\vec{m}^*}^{\vec{\epsilon}} = \left\{ \vec{m} : \frac{|m_i - m_i^*|}{m_i^*} \leq \epsilon_i, \vec{\epsilon} = [\epsilon_1 \; \epsilon_2 \; \ldots \; \epsilon_i \; \ldots \; \epsilon_r], i = 1, \ldots, r \right\}$$

where m_i is an integer.

3. Formulation of Optimal Hypercycle Network Design

The hypercycle network, as a massively parallel system, requires high throughput and low network latency to achieve maximum performance. In general, the average distance of the network is directly related to the network latency and throughput [2, 17] and the optimal network design is therefore achieved by minimizing the average distance. Since the node radix vector, $\vec{m}$, and connectivity vector $\vec{\rho}$ of an r-D hypercycle can assume only discrete values, the optimization will involve integer design variables. In addition, physical realizability of a network system imposes constraints on the design parameters. The formulation of a nonlinear integer programming problem for average distance is presented subsequently.

Average Distance: Given a source node $(\alpha)_{m_1 m_2 \ldots m_r} = \alpha_1 \alpha_2 \ldots \alpha_r$ and a destination node $(\beta)_{m_1 m_2 \ldots m_r} = \beta_1 \beta_2 \ldots \beta_r$, if $D_{\rho_i}(\beta_i, \alpha_i)$ denotes the shortest distance separating α and β in dimension i, then the Hamming distance between α and β is given by $H_D(\alpha, \beta) = \sum_{i=1}^{r} D_{\rho_j}(\beta_i, \alpha_i)$.

The average distance, δ, of a hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is defined as the normalized sum of the distances of all nodes in the graph with respect to a reference node. This can be computed by summing the distance of each node of $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ with respect to a reference node in each

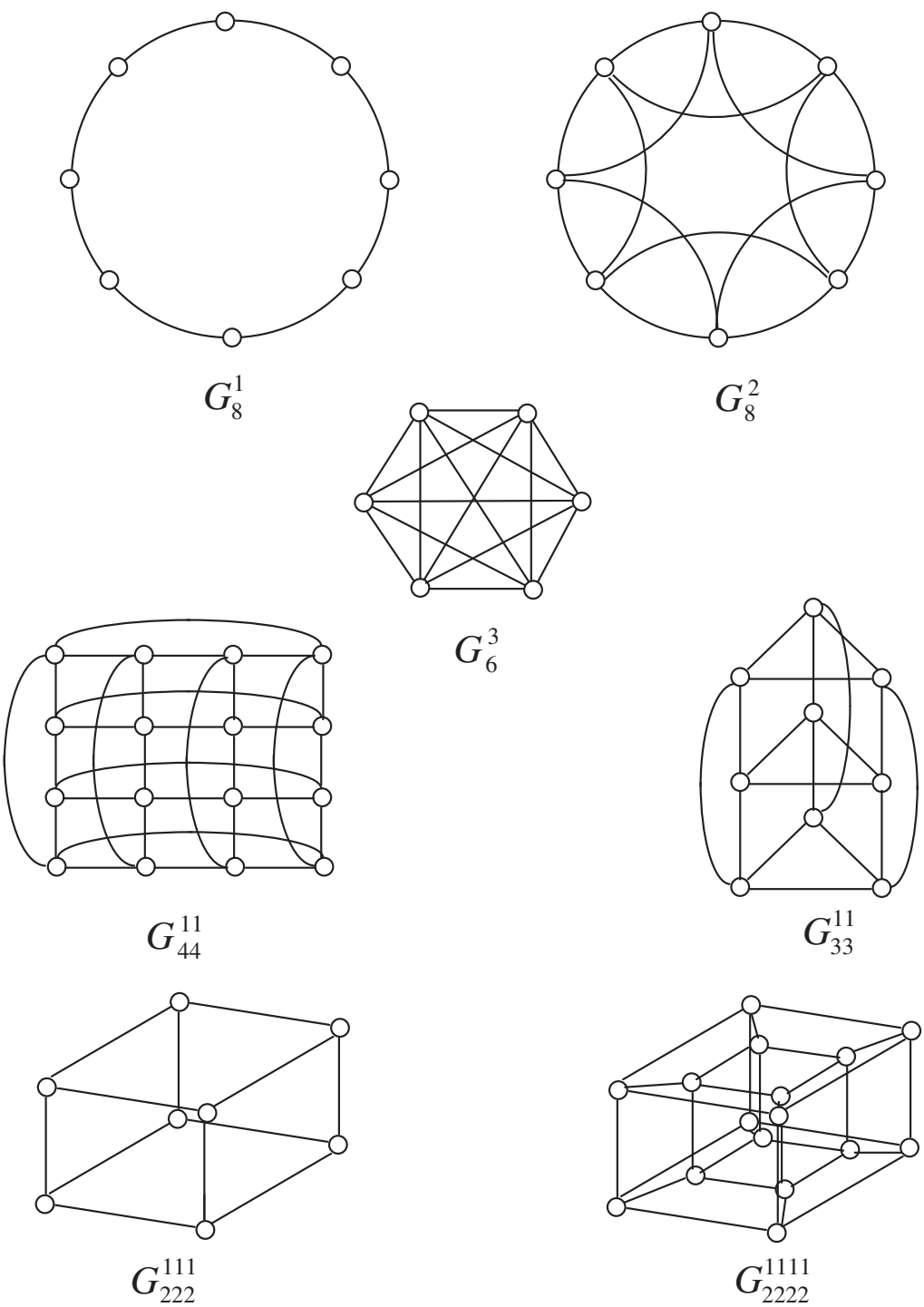

Figure 1 *Examples of hypercycle graphs*

dimension i, for $i = 1, \ldots, r$. Since hypercycles are symmetrical graphs by definition, it can be assumed without loss of generality that node $\alpha = \underbrace{(0,0,0\ldots,0)}_{r}$ is the reference node. Furthermore, let $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$ be any arbitrary node. If we consider the distance along dimension j between α and nodes $\beta' = \underbrace{(X, X, \ldots, \beta_j, X, X, \ldots X)}_{r}^{*}$, then there are $\prod_{i=1, i\neq j}^{r} m_i$ nodes having address β_j in the j^{th} dimension. Therefore the total distance along dimension j for all these β' nodes is given by $D_{\rho_j}(\beta_j, 0).\prod_{i=1, i\neq j}^{r} m_i$. Since $0 \leq \beta_j \leq m_j - 1$, the total distance in the j^{th} dimension between α and all nodes in $\mathcal{G}_m^{\rho}$ is given by $\sum_{\beta_j=0}^{m_j-1} D_{\rho_j}(\beta_j, 0)\ \prod_{i=1, i\neq j}^{r} m_i$. This is summed over r dimensions and the the total distance is given by $\sum_{j=1}^{r}\sum_{\beta_j=0}^{m_j-1} D_{\rho_j}(\beta_j, 0)\ \prod_{i=1, i\neq j}^{r} m_i$. This quantity, when divided by $M - 1$ (neglecting the reference node) gives the required average distance.

The expression for average distance is given as follows:

$$\delta = \frac{\sum_{j=1}^{r} S_j}{M-1} \tag{4}$$

where

$$S_j = f(m_j, \rho_j) \prod_{k=1, k\neq j}^{r} m_k \tag{5}$$

$$f(m_j, \rho_j) = \sum_{\beta_j=0}^{m_j-1} D_{\rho_j}(\beta_j, 0) \tag{6}$$

$$D_{\rho_j}(\beta_j, 0) = \begin{cases} \left\lceil \frac{\beta_j}{\rho_j} \right\rceil & \text{if } \beta_j \leq \lfloor \frac{m_j}{2} \rfloor \\ \left\lceil \frac{m_j - \beta_j}{\rho_j} \right\rceil & \text{otherwise} \end{cases} \tag{7}$$

The variable S_j denotes the distance of all nodes in the j^{th} dimension from the reference node. The variable $D_{\rho_j}(\beta_j, 0)$, defined by Eq. (7), gives the shortest distance in dimension j for node m_j with respect to the reference node and is computed using the *greedy strategy* outlined in [20]. Thus, the problem of determining optimal hypercycle network configurations which minimize the average distance can be formulated as follows:

Case 1: Given the total number of nodes M, the number of dimensions r and the connectivity sum C (which loosely corresponds to the degree of the network), find the node radix vector $\vec{m} = [m_1\ m_2\ \ldots\ m_r]$ and the connectivity vector $\vec{\rho} = [\rho_1\ \rho_2\ \ldots\ \rho_r]$ that define the graph with the minimum average distance.

$$\min_{w.r.t\ m_j,\ \rho_j} \delta(m_j,\ \rho_j) = \frac{1}{M-1} \sum_{j=1}^{r} S_j$$

*X denotes a don't care address digit

subject to

$$\begin{aligned} h_1 &= M - \prod_{j=1}^{r} m_j = 0 \\ h_2 &= C - \sum_{j=1}^{r} \rho_j = 0 \\ g_i &= 1 - \rho_i \leq 0 \quad (i = 1, \ldots, r) \\ g_{i+r} &= 2 - m_i \leq 0 \end{aligned}$$

The above general problem can also be specialized as the following two problems.

Case 2: Given the connectivity vector $\vec{\rho}$, r and M, find the node radix vector $\vec{m}$, that will minimize the average distance.

$$\min_{w.r.t\ m_j} \delta(m_j) = \frac{1}{M-1} \sum_{j=1}^{r} S_j$$

subject to

$$\begin{aligned} h_1 &= M - \prod_{j=1}^{r} m_j = 0 \\ h_{2+i} &= \rho_i = c_i \quad (i = 1, \ldots, r) \\ g_i &= 2 - m_i \leq 0 \end{aligned}$$

where c_i is some integer constant bounded from above by $\left\lfloor \frac{M}{2^r} \right\rfloor$.

Case 3: Given the node radix vector $\vec{m}$, r, and connectivity-sum C, optimize the average distance to determine the connectivity vector $\vec{\rho}$.

$$\min_{w.r.t\ \rho_j} \delta(\rho_j) = \frac{1}{M-1} \sum_{j=1}^{r} S_j$$

subject to

$$\begin{aligned} h_i &= m_i = c_i \quad (i = 1, \ldots, r) \\ h_{r+i} &= C - \prod_{j=1}^{r} \rho_j = 0 \end{aligned}$$

4. Optimization of the average distance and diameter (Case 1)

In this section, we examine the optimization problem as presented in Case 1 earlier and establish the structure of hypercycles which have minimum average distance (diameter) given a fixed number of nodes, connectivity sum and assuming that the number of dimensions is a given constant. Since the objective functions are discrete, we compute the optimum in two stages:

- Firstly, we establish continuous upper and lower bounds and determine the optimum point with regard to those bounds.
- Secondly, small perturbations are applied to the optimal point derived for the bounds and we prove that these perturbations lead to non-optimal points for the original discrete function.

4.1. Bounds for average distance and diameter

Lemma 1 *The average distance function δ given by Eq. (4) is bounded from above and below by*

$$\hat{\delta} = \frac{M}{(M-1)} \left(\frac{m_1}{4\rho_1} + \sum_{j=2}^{r} \frac{m_j}{4\rho_j} + r \right) \quad \textit{and} \tag{8}$$

$$\check{\delta} = \frac{M}{(M-1)} \left(\frac{m_1}{4\rho_1} + \sum_{j=2}^{r} \frac{m_j}{4\rho_j} - \frac{r}{4} \right) \tag{9}$$

respectively

Proof. Refer to Appendix A. ■

Lemma 2 *The cost optimal point for the upper and lower bounds of the average distance function, namely, $\hat{\delta}$ and $\check{\delta}$ as specified by Case 1 is $m_i = \sqrt[r]{M}$, $\rho_i = \frac{C}{r}$, $i = 1, \ldots, r$ where r, M and C are given constants.*

Proof.

Substituting $m_1 = M/(m_t \prod_{j=2, j\neq t}^{r} m_j)$ in Eq. (8) where $2 \leq t \leq r$, we obtain

$$\hat{\delta} = \frac{M}{M-1} \left(\frac{M}{4\rho_1 m_t \prod_{j=2, j\neq t}^{r} m_j} + \frac{m_t}{4\rho_t} + \sum_{j=2, j\neq t}^{r} \frac{m_j}{4\rho_j} + r \right) \tag{10}$$

To find the minimum of $\hat{\delta}$, the first order necessary condition given by $\bigtriangledown\hat{\delta}(m^*, \rho^*) = 0$ has to be satisfied by the optimal point. Computing the derivative of $\hat{\delta}$ with respect to m_t, we have

$$\frac{\partial\hat{\delta}}{\partial m_t} = \frac{M}{4(M-1)} \left(\frac{-M}{m_t^2 \rho_1 \prod_{j=2, j\neq t}^{r} m_j} + \frac{1}{\rho_t} \right) = 0$$

i.e.,

$$\frac{m_1}{m_t} = \frac{\rho_1}{\rho_t} \tag{11}$$

and

$$\frac{\partial^2\hat{\delta}}{\partial m_t^2} = \frac{M}{4(M-1)} \left(\frac{2m_1}{\rho_1 m_t^2} \right) > 0, \quad \forall m_t \geq 2$$

Substituting $\rho_1 = C - \rho_t - \sum_{j=2, j\neq t}^{r} \rho_j$ in Eq. (8), we obtain

$$\hat{\delta} = \frac{M}{M-1}\left(\frac{m_1}{4(C-\rho_t-\sum_{j=2,j\neq t}^{r}\rho_j)} + \frac{m_t}{4\rho_t} + \sum_{j=2,j\neq t}^{r}\frac{m_j}{4\rho_j} + r\right) \tag{12}$$

Taking the derivative of Eq. (12) with respect to ρ_t, $2 \leq t \leq r$, yields

$$\frac{\partial\hat{\delta}}{\partial\rho_t} = \frac{m_1}{4\rho_1^2} - \frac{m_t}{4\rho_t^2} = 0$$

i.e.,

$$\frac{m_1}{m_t} = \frac{\rho_1^2}{\rho_t^2} \tag{13}$$

and

$$\frac{\partial^2\hat{\delta}}{\partial\rho_t^2} = \frac{M}{(M-1)}\left(\frac{m_t}{2\rho_t^3}\right) > 0, \quad \forall\rho_t \geq 1$$

From Eq. (11) and (13), we obtain

$$\frac{\rho_1}{\rho_t} = \frac{\rho_1^2}{\rho_t^2} \quad \text{and}$$

$$\frac{\rho_1}{\rho_t}\left(\frac{\rho_1}{\rho_t} - 1\right) = 0$$

which leads to

$$\rho_1 = \rho_t$$
$$m_1 = m_t$$

It therefore follows that $\hat{\delta}$ is a minimum when $\rho_1 = \rho_2 = \ldots = \rho_r = \frac{C}{r}$ and $m_1 = m_2 = \ldots = m_i = \ldots = m_r = \sqrt[r]{M}$, $i = 1, \ldots, r$. In a similar manner, it can be shown that the minimum of $\check{\delta}$ given by Eq. (9) is identical to that of $\hat{\delta}$. ■

Lemma 3 *The diameter of the hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is bounded from above by $\hat{k} = \sum_{i=1}^{r}\frac{m_i}{2\rho_i} + r$.*

Proof. From Eq. (2), $k = \sum_{i=1}^{r}\left\lceil\frac{\lfloor\frac{m_i}{2}\rfloor}{\rho_i}\right\rceil$. Hence $k \leq \sum_{i=1}^{r}\left\lceil\frac{\frac{m_i}{2}}{\rho_i}\right\rceil \leq \sum_{i=1}^{r}\left(\frac{m_i}{2\rho_i}+1\right) = \sum_{i=1}^{r}\frac{m_i}{2\rho_i} + r$. Therefore $\hat{k} = \sum_{i=1}^{r}\frac{m_i}{2\rho_i} + r$. ■

Lemma 4 *The diameter of the hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is bounded from below by $\check{k} = \sum_{i=1}^{r}\frac{m_i}{2\rho_i} - r$.*

Proof. From $k = \sum_{i=1}^{r}\left\lceil\frac{\lfloor\frac{m_i}{2}\rfloor}{\rho_i}\right\rceil$ we have $k \geq \sum_{i=1}^{r}\left\lceil\frac{\frac{m_i}{2}-1}{\rho_i}\right\rceil \geq \sum_{i=1}^{r}\left(\frac{m_i}{2\rho_i} - \frac{1}{\rho_i}\right) \geq \sum_{i=1}^{r}\frac{m_i}{2\rho_i} - r$. Therefore $\check{k} = \sum_{i=1}^{r}\frac{m_i}{2\rho_i} - r$. ■

Lemma 5 *The cost optimal point for the upper and lower bounds of the diameter, $\hat{k}$ and $\check{k}$ is given by $m_i = \sqrt[r]{M}$, $\rho_i = \frac{C}{r}$, $i = 1, \ldots, r$ where r, M and C are given constants.*

Proof. From Lemma 3, we know that $\hat{k} = \sum_{i=1}^{r} \frac{m_i}{2\rho_i} + r = \hat{k} = \frac{m_1}{2\rho_1} + \sum_{j=2}^{r} \frac{m_j}{2\rho_j} + r$. Substituting, $m_1 = M/(m_t \prod_{j=2, j\neq t}^{r} m_j)$, $2 \leq t \leq r$, we get

$$\hat{k} = \frac{M}{2\rho_1 m_t \prod_{j=2, j\neq t}^{r} m_j} + \frac{m_t}{2\rho_t} + \sum_{j=2, j\neq t}^{r} \frac{m_j}{2\rho_j} + r$$

Evaluating the first order derivative of $\hat{k}$ w.r.t m_t, we obtain

$$\frac{\partial \hat{k}}{\partial m_t} = \frac{-M}{2m_t^2 \rho_1 \prod_{j=2, j\neq t}^{r} m_j} + \frac{1}{2\rho_t} = 0 \quad \text{and hence}$$

$$\frac{m_1}{m_t} = \frac{\rho_1}{\rho_t} \tag{14}$$

With $\rho_1 = C - \rho_t - \sum_{j=2, j\neq t}^{r} \rho_j$, $2 \leq t \leq r$, we have

$\hat{\delta} = \left(\frac{m_1}{2\left(C - \rho_t - \sum_{j=2, j\neq t}^{r} \rho_j\right)} + \frac{m_t}{2\rho_t} + \sum_{j=2, j\neq t}^{r} \frac{m_j}{2\rho_j} + r \right)$ and evaluating the derivative of $\hat{k}$ with respect to ρ_t yields

$$\frac{\partial \hat{\delta}}{\partial \rho_t} = \frac{m_1}{2\rho_1^2} - \frac{m_t}{2\rho_t^2} = 0 \tag{15}$$

From Eqs. (14) and (15), we obtain $m_1 = m_t$, $\rho_1 = \rho_t$ and hence $k_1 = k_t$, for $t = 2, \ldots, r$. Likewise an identical result can be obtained for $\check{k}$ as well. ■

4.2. Perturbation Analysis

In this section, we examine whether the optimal point obtained for the bounds in Section 4.1 remain optimal for the original discrete functions, namely the average distance and diameter. We establish the optimality by calculating the value of the discrete function in the neighbourhood of the optimal point and showing that this value is always larger than the value at the optimal point. Before proceeding further, we state Lemmas 6, 7 and 8 which establish some intermediate computation results and which will be used in establishing our main result. The proofs of these Lemmas are given in Appendix B

Lemma 6

$$f(m, \rho) = \begin{cases} 2\sum_{i=1}^{\frac{m-1}{2}} \left\lceil \frac{i}{\rho} \right\rceil & \text{if } m \text{ is odd} \\ 2\sum_{i=1}^{\frac{m}{2}} \left\lceil \frac{i}{\rho} \right\rceil - \left\lceil \frac{m}{2\rho} \right\rceil & \text{if } m \text{ is even} \end{cases} \tag{16}$$

Lemma 7

$$\sum_{i=1}^{n} \left\lceil \frac{i}{\rho} \right\rceil = \frac{n}{2} + \frac{n^2}{2\rho} + \frac{X(\rho - X)}{2\rho} \quad \text{where } X = n \bmod \rho \tag{17}$$

Lemma 8 *For* $1 \leq \rho \leq \lfloor \frac{m}{2} \rfloor$ *and* $\lfloor \frac{m}{2} \rfloor \bmod \rho = 0$, *then* $f(m,\rho)$ *can be expressed as*

$$f(m,\rho) = \begin{cases} \frac{m-1}{2} + \frac{(m-1)^2}{4\rho} & \text{if } m \text{ is odd} \\ \\ \frac{m}{2} + \frac{m(m-2)}{4\rho} & \text{if } m \text{ is even} \end{cases} \tag{18}$$

Theorem 1 *A balanced hypercycle* $\mathcal{G}^{\vec{\rho}}_{\vec{m}}$ *is cost-optimal for average distance when* m_i *and* ρ_i *are integers such that* $m_i = \sqrt[r]{M}$, $\rho_i = \frac{C}{r}$, *and* $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i = 0$, *for* $i = 1, \ldots, r$, *where the dimension* r, *the total number of nodes* M *and the connectivity-sum* C *are given integer constants.*

Proof. For the balanced hypercycle, let $m_j = m$ and $\rho_j = \rho$, $(j = 1, \ldots, r)$. If δ_b denotes its corresponding average distance, then we can express

$$\begin{aligned} \delta_b &= \frac{\sum_{j=1}^{r} S_j}{M-1} \\ &= \frac{rS}{M-1} \end{aligned} \tag{19}$$

where

$$\begin{aligned} S &= f(m,\rho) \prod_{k=1, k\neq j}^{r} m_k \\ &= m^{r-1} f(m,\rho) \end{aligned} \tag{20}$$

Substituting Eq. (20) in Eq. (19), the average distance for the balanced hypercycle is given by

$$\delta_b = \frac{r m^{r-1} f(m,\rho)}{M-1} \tag{21}$$

Suppose we move away from $m_1 = m_2 = \ldots = m_r = m$ to a point m^* given by $\underbrace{m, \ldots, m+v, \ldots, m-\lambda, \ldots, m}_{r}$ where λ and v are integer variables such that $m^r = (m+v)(m-\lambda)m^{r-2} \Rightarrow m^2 = (m+v)(m-\lambda)$ then $mv = \lambda(m+v)$ which implies

$$\lambda = \frac{v}{1+\frac{v}{m}} \tag{22}$$

and

$$v - \lambda = \frac{v\lambda}{m} > 0 \quad \text{(assuming that both } v, \lambda \neq 0) \tag{23}$$

Observe that $0 \leq \lambda \leq m-2$ and $0 \leq v \leq \frac{m}{2}(m-2)$ when $m \geq 2$. It is worth noting that $v \neq 1$ since otherwise $\lambda = \frac{m}{m+1} < 1$. Thus there exists no positive integer λ which would satisfy Eq. (22) when $v = 1$. By a similar token, $\lambda \neq 1$.

The average distance calculated at the perturbed point m^* is given by

$$\begin{aligned} \delta' = \frac{1}{M-1} \Big[&(r-2)m^{r-3}m^2 f(m,\rho) + m^{r-2}(m+v) f(m-\lambda,\rho) \\ &+ m^{r-2}(m-\lambda) f(m+v,\rho) \Big] \end{aligned} \tag{24}$$

The difference between δ' and δ_b is given by

$$\begin{aligned}\delta' - \delta_b &= \frac{1}{M-1}\Big[(r-2).m^{r-3}(m+v)(m-\lambda).f(m,\rho) + m^{r-2}(m+v)f(m-\lambda,\rho) \\ &\qquad +m^{r-2}(m-\lambda).f(m+v,\rho) - r.m^{r-1}f(m,\rho)\Big] \\ &= \frac{m^{r-2}}{M-1}\left[(m+v)f(m-\lambda,\rho) + (m-\lambda)f(m+v,\rho) - 2mf(m,\rho)\right] \qquad (25)\end{aligned}$$

If δ_b is indeed a true minimal point, then any one step perturbation will result in a value of δ' being larger than δ_b. i.e., $\delta' - \delta_b \geq 0$ for a given ρ $\forall v, \lambda \geq 0$. In the following discussion, we shall analyze Eq. (25) when $X = \lfloor \frac{m}{2} \rfloor \bmod \rho = 0$. Observe that Eqs. (88) and (90) given in Appendix B precisely define $f(m,\rho)$ when $X = 0$. Consider the following cases of m.

1. m **- Odd**

When m is odd, the factors of m^2, namely $(m-\lambda)$ and $(m+v)$ are also odd. Hence, substituting Eq. (88) for $f(m-\lambda,\rho)$ and $f(m+v,\rho)$ in Eq. (25), we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &\geq m^{r-2}\left[(m+v)\left[\frac{(m-\lambda-1)}{2} + \frac{(m-\lambda-1)^2}{4\rho}\right]\right. \\ &\qquad \left. +(m-\lambda)\left[\frac{(m+v-1)}{2} + \frac{(m+v-1)^2}{4\rho}\right] - 2mf(m,\rho)\right] \\ &= m^{r-2}\left(\frac{(m^2-m-v)}{2} + \frac{(m-\lambda-1)(m^2-m-v)}{4\rho}\right. \\ &\qquad \left. +\frac{(m^2-m+\lambda)}{2} + \frac{(m+v-1)(m^2-m+\lambda)}{4\rho} - 2mf(m,\rho)\right) \\ &= m^{r-2}\left[m^2 - m + \left(\frac{(m-1)^2}{4\rho} - \frac{1}{2}\right)(v-\lambda) + \frac{2v\lambda}{4\rho} + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m}{2\rho}\right. \\ &\qquad \left. -2mf(m,\rho)\right]\end{aligned}$$

Similarly, substituting Eq. (18) for $f(m,\rho)$ in the above, we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &\geq m^{r-2}\left[m^2 - m + \left(\frac{(m-1)^2}{4\rho} - \frac{1}{2}\right)(v-\lambda) + \frac{2v\lambda}{4\rho} + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m}{2\rho}\right. \\ &\qquad \left. -2m\left(\frac{m-1}{2} + \frac{(m-1)^2}{4\rho}\right)\right] \\ &= m^{r-2}\left[m^2 - m + \left(\frac{(m-1)^2}{4\rho} - \frac{1}{2}\right)(v-\lambda) + \frac{2v\lambda}{4\rho} + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m}{2\rho}\right. \\ &\qquad \left. -m(m-1) - \frac{m(m-1)^2}{2\rho}\right] \\ &= m^{r-2}\left[m^2 - m + \left(\frac{(m-1)^2}{4\rho} - \frac{1}{2}\right)(v-\lambda) + \frac{2v\lambda}{4\rho} + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m}{2\rho}\right.\end{aligned}$$

$$
\begin{aligned}
& -m^2 + m - \frac{m^3}{2\rho} + \frac{m^2}{\rho} - \frac{m}{2\rho}\Bigg] \\
= \quad & m^{r-2}\left[\left(\frac{(m-1)^2}{4\rho} - \frac{1}{2}\right)(\upsilon - \lambda) + \frac{2\upsilon\lambda}{4\rho}\right] \geq 0 \qquad (26)
\end{aligned}
$$

since $(\upsilon - \lambda) = \frac{\upsilon\lambda}{m} = \frac{\upsilon^2}{m+\upsilon} \geq 0$ and $\frac{(m-1)^2}{4\rho} - \frac{1}{2} \geq \frac{m-2}{2} > 0$ for $m \geq 3$ and $1 \leq \rho \leq \frac{m-1}{2}$.

2. m **- Even**

When m is even, the factors of m^2, namely, $m - \lambda$ and $m + \upsilon$ can assume 3 possible combinations depending on whether λ and υ are even or odd.

(a) υ *and* λ *- Even*

In this case,$m - \lambda$ and $m + \upsilon$ are even. Substituting Eq. (90) for $f(m - \lambda, \rho)$ and $f(m + \upsilon, \rho)$ in Eq. (25), we obtain

$$
\begin{aligned}
(M-1)(\delta' - \delta_b) \quad \geq \quad & m^{r-2}\Bigg[(m+\upsilon)\left[\frac{(m-\lambda)}{2} + \frac{(m-\lambda)(m-\lambda-2)}{4\rho}\right] \\
& +(m-\lambda)\left[\frac{(m+\upsilon)}{2} + \frac{(m+\upsilon)(m+\upsilon-2)}{4\rho}\right] - 2mf(m,\rho)\Bigg] \\
= \quad & m^{r-2}\left[\frac{m^2}{2} + \frac{m^2(m-\lambda-2)}{4\rho} + \frac{m^2}{2} + \frac{m^2(m+\upsilon-2)}{4\rho} - 2mf(m,\rho)\right] \\
= \quad & m^{r-2}\left[m^2 + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m^2}{4\rho}(\upsilon - \lambda) - 2mf(m,\rho)\right]
\end{aligned}
$$

Substituting for $f(m, \rho)$ (m being even) from Eq. (18) in the above, we obtain

$$
\begin{aligned}
\delta' - \delta_b \quad \geq \quad & m^{r-2}\left[m^2 + \frac{m^3}{2\rho} - \frac{m^2}{\rho} + \frac{m^2(\upsilon-\lambda)}{4\rho} - 2m\left(\frac{m}{2} + \frac{m(m-2)}{4\rho}\right)\right] \\
\geq \quad & \left(\frac{m^r}{4\rho}\right)(\upsilon - \lambda) \geq 0 \qquad (27)
\end{aligned}
$$

since $\upsilon > \lambda$.

(b) υ *- odd and* λ *- even*

In this case, substituting Eqs. (88) and (90) for $(m+\upsilon, \rho)$ and $f(m-\lambda, \rho)$ respectively, we obtain

$$
\begin{aligned}
(M-1)(\delta' - \delta_b) \quad \geq \quad & m^{r-2}\Bigg[(m+\upsilon)\left[\frac{(m-\lambda)}{2} + \frac{(m-\lambda)(m-\lambda-2)}{4\rho}\right] \\
& +(m-\lambda)\left[\frac{(m+\upsilon-1)}{2} + \frac{(m+\upsilon-1)^2}{4\rho}\right] - 2mf(m,\rho)\Bigg] \\
= \quad & m^{r-2}\Bigg[\frac{m^2}{2} + \frac{m^2(m-\lambda-2)}{4\rho} + \frac{(m^2-m+\lambda)}{2} \\
& +\frac{(m+\upsilon-1)(m^2-m+\lambda)}{4\rho} - 2mf(m,\rho)\Bigg]
\end{aligned}
$$

$$= m^{r-2}\left[m^2+\frac{m^3}{2\rho}+\frac{v\lambda}{4\rho}+\frac{m-\lambda}{4\rho}+\frac{\lambda}{2}+\frac{(v-\lambda)m(m-1)}{4\rho} -\frac{m}{2}-\frac{m^2}{\rho}-2mf(m,\rho)\right]$$

Substituting $f(m,\rho)$ given by Eq. (90) in the above, we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &\geq m^{r-2}\left[m^2+\frac{m^3}{2\rho}+\frac{v\lambda}{4\rho}+\frac{m-\lambda}{4\rho}+\frac{\lambda}{2}+\frac{(v-\lambda)m(m-1)}{4\rho} -\frac{m}{2}-\frac{m^2}{\rho}-2m\left(\frac{m}{2}+\frac{m(m-2)}{4\rho}\right)\right]\\ &= m^{r-2}\left[\frac{(v-\lambda)m(m-1)}{4\rho}+\frac{v\lambda}{4\rho}+\frac{m-\lambda}{4\rho}+\frac{\lambda}{2}-\frac{m}{2}\right]\\ &= m^{r-2}\left[\frac{m^2(v-\lambda)}{4\rho}+\frac{m-\lambda}{4\rho}+\frac{\lambda}{2}-\frac{m}{2}\right]\geq 0 \qquad \forall m,\rho \quad (28)\end{aligned}$$

Observe that $\frac{m^2(v-\lambda)}{4\rho}-\frac{m}{2}=\frac{m}{2}\left(\frac{m(v-\lambda)}{2\rho}-1\right)$. Since $v-\lambda\geq 1$ and $m\geq 2\rho$, it implies that $m(v-\lambda)\geq 2\rho$ and $\frac{m(v-\lambda)}{2\rho}-1\geq 0$.

(c) v - even and λ - odd

In a similar manner as above, substituting Eqs. (88) and (90) for $f(m-\lambda,\rho)$ and $(m+v,\rho)$ respectively, we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &\geq m^{r-2}\left[(m+v)\left[\frac{(m-\lambda-1)}{2}+\frac{(m-\lambda-1)^2}{4\rho}\right] +(m-\lambda)\left[\frac{(m+v)}{2}+\frac{(m+v)(m+v-2)}{4\rho}\right]-2mf(m,\rho)\right]\\ &= m^{r-2}\left[\frac{(m^2-m-v)}{2}+\frac{(m-\lambda-1)(m^2-m-v)}{4\rho}+\frac{m^2}{2} +\frac{m^2(m+v-2)}{4\rho}-2mf(m,\rho)\right]\\ &= m^{r-2}\left[m^2+\frac{m^3}{2\rho}+\frac{(v-\lambda)m(m-1)}{4\rho}-\frac{m^2}{\rho}+\frac{m+v}{\rho} -\frac{m+v}{2}+\frac{v\lambda}{4\rho}-2mf(m,\rho)\right]\end{aligned}$$

Substituting $f(m,\rho)$ given by Eq. (90) in the above, we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &\geq m^{r-2}\left[m^2+\frac{m^3}{2\rho}+\frac{(v-\lambda)m(m-1)}{4\rho}-\frac{m^2}{\rho}+\frac{m+v}{\rho} -\frac{m+v}{2}+\frac{v\lambda}{4\rho}-2m\left(\frac{m}{2}+\frac{m(m-2)}{4\rho}\right)\right]\end{aligned}$$

$$= m^{r-2}\left[\frac{(\upsilon-\lambda)m(m-1)}{4\rho}+\frac{m+\upsilon}{\rho}-\frac{m+\upsilon}{2}+\frac{\upsilon\lambda}{4\rho}\right]$$
$$= m^{r-2}\left[\frac{m^2(\upsilon-\lambda)}{4\rho}+\frac{m+\upsilon}{\rho}-\frac{m+\upsilon}{2}\right] \quad (29)$$

since $\upsilon-\lambda=\frac{\upsilon\lambda}{m}$.

To prove $\delta'-\delta_b \geq 0$ as given Eq. (29) for $1 \leq \rho \leq \frac{m-1}{2}$, we consider the following cases:

1. $1 \leq \rho \leq \frac{m-\lambda-1}{2}$
   Substituting the upper bound of ρ, namely, $\rho=\frac{m-\lambda-1}{2}$ in Eq. (29), we obtain
   $$(M-1)(\delta'-\delta_b) \geq m^{r-2}\left[\frac{m^2(\upsilon-\lambda)}{2(m-\lambda-1)}+\frac{2(m+\upsilon)}{m-\lambda-1}-\frac{m+\upsilon}{2}\right]$$
   $$= m^{r-2}\left[\frac{m^2(\upsilon-\lambda)+(m+\upsilon)-m^2+(m+\upsilon)}{2(m-\lambda-1)}\right]$$
   $$= m^{r-2}\left[\frac{m^2(\upsilon-\lambda-1)+2(m+\upsilon)}{2(m-\lambda-1)}\right] \geq 0$$
   since $\upsilon-\lambda \geq 1$ when $\upsilon,\lambda \neq 0$.
2. $\frac{m-\lambda+1}{2} \leq \rho \leq \frac{m}{2}$
   Since $\rho \geq \frac{m-\lambda+1}{2} \geq \lfloor\frac{m-\lambda}{2}\rfloor$, it implies that $f(m-\lambda,\rho)=m-\lambda-1$ since it fully connected. Hence
   $$(M-1)(\delta'-\delta_b) \geq m^{r-2}\left[(m+\upsilon)(m-\lambda-1)+(m-\lambda)\frac{(m+\upsilon-1)}{2}\right.$$
   $$\left.+(m-\lambda)\frac{(m+\upsilon-1)^2}{4\rho}-2mf(m,\rho)\right]$$
   $$= m^{r-2}\left[m^2-m-\upsilon+\frac{m^2}{2}+\frac{m^2(m+\upsilon-2)}{4\rho}-2mf(m,\rho)\right]$$
   $$= m^{r-2}\left[m^2-m-\upsilon+\frac{m^2}{2}+\frac{m^2(m+\upsilon-2)}{4\rho}\right.$$
   $$\left.-2m\left(\frac{m}{2}+\frac{m(m-2)}{4\rho}\right)\right]$$
   $$= m^{r-2}\left[m^2\left(\frac{\upsilon+2}{4\rho}\right)-(m+\upsilon)+\frac{m^2}{2}-\frac{m^3}{4\rho}\right]$$
   Substituting the lower and upper bounds of ρ in the above equation, we obtain,
   $$(M-1)(\delta'-\delta_b) \geq m^{r-2}\left[m^2\left(\frac{\upsilon+2}{2m}\right)-(m+\upsilon)+\frac{m^2}{2}-\frac{m^3}{2(m-\lambda+1)}\right]$$
   $$= m^{r-2}\left[\frac{m(\upsilon+2)}{2}-(m+\upsilon)+\frac{m^2}{2}\left(1-\frac{1}{m-\lambda+1}\right)\right]$$

$$= \quad m^{r-2}\left[\upsilon\left(\frac{m}{2}-1\right)+\frac{m^2}{2}\left(1-\frac{1}{m-\lambda+1}\right)\right] \geq 0$$

since $m \geq 2$.

■

Theorem 2 *The balanced hypercycle $\mathcal{G}^{\vec{\rho}}_{\vec{m}}$ has a cost optimal diameter when $m_i = \sqrt[r]{M}$ and $\rho_i = \frac{C}{r}$ such that $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i = 0$, $\forall i \in \{1, 2, \ldots, r\}$ where C is a given constant.*

Proof. For the balanced hypercycle, the diameter k_b is given by

$$k_b = r\left\lceil \frac{\lfloor \frac{m}{2} \rfloor}{\rho} \right\rceil \tag{30}$$

where $m = \sqrt[r]{M}$ and $\rho = \frac{C}{r}$.

In an analysis similar to the average distance, we consider a 2-D perturbation from the balanced point m when any two dimension bases change to $(m+\upsilon)$ and $(m-\lambda)$ such that $m^2 = (m+\upsilon)(m-\lambda)$. Then, the new diameter k' is given by

$$\begin{aligned} k' &= (r-2)\left\lceil \frac{\lfloor \frac{m}{2} \rfloor}{\rho} \right\rceil + \left\lceil \frac{\lfloor \frac{m+\upsilon}{2} \rfloor}{\rho} \right\rceil + \left\lceil \frac{\lfloor \frac{m-\lambda}{2} \rfloor}{\rho} \right\rceil \\ k'-k_b &= \left\lceil \frac{\lfloor \frac{m+\upsilon}{2} \rfloor}{\rho} \right\rceil + \left\lceil \frac{\lfloor \frac{m-\lambda}{2} \rfloor}{\rho} \right\rceil - 2\left\lceil \frac{\lfloor \frac{m}{2} \rfloor}{\rho} \right\rceil \end{aligned} \tag{31}$$

Since $X = \lfloor \frac{m}{2} \rfloor \bmod \rho = 0$, we have

$$\begin{aligned} k'-k_b &= \left\lceil \frac{\lfloor \frac{m+\upsilon}{2} \rfloor}{\rho} \right\rceil + \left\lceil \frac{\lfloor \frac{m-\lambda}{2} \rfloor}{\rho} \right\rceil - \frac{2\lfloor \frac{m}{2} \rfloor}{\rho} \\ &\geq \frac{\lfloor \frac{m+\upsilon}{2} \rfloor}{\rho} + \frac{\lfloor \frac{m-\lambda}{2} \rfloor}{\rho} - \frac{2\lfloor \frac{m}{2} \rfloor}{\rho} \end{aligned} \tag{32}$$

Let us analyze Eq. (32) by considering the following cases.

Case 1: m - even

1. Both $m+\upsilon$ and $m-\lambda$ are even

$$\begin{aligned} k'-k_b &\geq \frac{1}{\rho}\left(\frac{m+\upsilon}{2}+\frac{m-\lambda}{2}-m\right) \\ &\geq \frac{\upsilon-\lambda}{2\rho} \geq 0 \end{aligned}$$

2. $m+\upsilon$ is even, $m-\lambda$ is odd and vice versa

$$\begin{aligned} k'-k_b &\geq \frac{1}{\rho}\left(\frac{m+\upsilon}{2}+\frac{m-\lambda}{2}-\frac{1}{2}-m\right) \\ &\geq \frac{\upsilon-\lambda-1}{2\rho} \geq 0 \quad \text{(from Eq. (23))} \end{aligned}$$

Case 2: m **- odd**

It implies that both $m+\upsilon$ and $m-\lambda$ are odd.

$$\begin{aligned} k' - k_b &\geq \frac{1}{\rho}\left(\frac{m+\upsilon-1}{2} + \frac{m-\lambda-1}{2} - (m-1)\right) \\ &= \frac{\upsilon-\lambda}{2\rho} > 0 \end{aligned}$$

■

When $X \neq 0$, it becomes mathematically intractable and difficult to establish $\delta' - \delta_b \geq 0$ (Eq. (25)) $\forall$ υ and λ owing to the high degree of complexity in Eqs. (87) and (89). By applying 2-D perturbations, we have shown that graphs (for which $\lfloor \frac{m}{2} \rfloor \bmod \rho = 0$) lead to larger average distance and diameter compared to the balanced structure. The main question that we need to answer is whether this minimum is indeed optimum when $X \neq 0$. As a matter of fact, one can find cases for which such perturbations lead to better optimal points than the balanced structure. In other words, the smallest average distance (diameter) is not always achieved when all the factors of the node radix vector $\vec{m}$ are equal.

Theorem 3 *When the number of nodes M, dimension r and connectivity-sum C are given constants, balanced hypercycles do not necessarily exhibit minimum average distance (diameter) when $m_i = m = \sqrt[r]{M}$, $\rho_i = \rho = \frac{C}{r}$ and $\lfloor \frac{m}{2} \rfloor \bmod \rho \neq 0$.*

Proof. This theorem can be proved through a counter example. Let $M = 144$, $r = 2$ and $C = 8$. Applying the result of Theorem 1, we obtain the balanced hypercycle $\mathcal{G}_{12}^{\,4\ 4}{}_{12}$ whose average distance is 2.349650. Observe that in this case $\lfloor \frac{m}{2} \rfloor \bmod \rho \neq 0$ while the hypercycle $\mathcal{G}_{9}^{\,4\ 4}{}_{16}$ has an average distance of 2.279720. Similarly the diameter of $\mathcal{G}_{12}^{\,4\ \ 4}{}_{12}$ is 4 while that of $\mathcal{G}_{9}^{\,4\ 4}{}_{16}$ is 3. ■

In fact one can find families of hypercycles that present smaller average distance (diameter) compared to the balanced hypercycle when $\lfloor \frac{m}{2} \rfloor \bmod \rho \neq 0$ and this will be discussed shortly in Theorem 4. Before proceeding further, we present Lemmas 9-15 whose proofs are given in Appendix C.

Lemma 9 *If $m = \rho(\rho-1)$, then $f(m,\rho) = \frac{\rho}{4}(\rho^2-2)$ where ρ is even.*

Lemma 10 *If $m+\upsilon = \rho^2$, then $f(m+\upsilon,\rho) = \frac{\rho}{4}(\rho^2+2\rho-2)$ where υ is a positive integer and ρ being even.*

Lemma 11 *If $m-\lambda = (\rho-1)^2$, then $f(m-\lambda,\rho) = \frac{\rho^2}{4}(\rho-2)$ where λ is a positive integer and ρ being even.*

Lemma 12 *If $m = (\rho-1)(\rho-2)$, then $f(m,\rho) = \frac{(\rho-2)(\rho^2-2\rho+2)}{4}$ where ρ is even.*

Lemma 13 *If $m-\lambda = (\rho-2)^2$, then $f(m-\lambda,\rho) = \frac{(\rho-2)(\rho^2-4\rho+6)}{4}$ where λ is a positive integer and ρ being even.*

Lemma 14 *If* $m = \rho(\rho+1)$, *then* $f(m,\rho) = \frac{(\rho+2)\left(\rho^2+2\rho-2\right)}{4}$ *where* ρ *is even.*

Lemma 15 *If* $m + v = (\rho+1)^2$, *then* $f(m+v,\rho) = \frac{\rho(\rho+2)(\rho+4)}{4}$ *where* v *is a positive integer and* ρ *being even.*

Theorem 4 *The following families of non-balanced 2-D hypercycles* $\mathcal{G}_{m_1 m_2}^{\rho\ \rho}$ *are cost-optimal for average distance compared to the balanced hypercycle* $\mathcal{G}_{mm}^{\rho\ \rho}$*:*

Family 1*:* ρ *is even and* $\rho \geq 4$

$$\begin{aligned} m &= \rho(\rho-1) \\ m_1 &= (\rho-1)^2 \\ m_2 &= \rho^2 \end{aligned}$$

Family 2*:* ρ *is even and* $\rho \geq 6$

$$\begin{aligned} m &= (\rho-2)(\rho-1) \\ m_1 &= (\rho-2)^2 \\ m_2 &= (\rho-1)^2 \end{aligned}$$

Family 3*:* ρ *is even and* $\rho \geq 4$

$$\begin{aligned} m &= \rho(\rho+1) \\ m_1 &= \rho^2 \\ m_2 &= (\rho+1)^2 \end{aligned}$$

Proof.

Family 1: Observe that $M = m^2 = m_1 m_2 = (m-\lambda)(m+v)$ and hence $m-\lambda = (\rho-1)^2$ and $m+v = \rho^2$. The connectivity constraint is satisfied for $m-\lambda$ since $2\rho < \rho^2-2\rho+1$ which implies that $\rho^2 - 4\rho + 1 > 0$ for $\rho \geq 4$. Substituting $r = 2$ and Eqs (91), (92) and (93) in Eq. (25), the difference $\delta' - \delta_b$ between the non-balanced and balanced hyperycles can be computed as

$$\begin{aligned} (M-1)(\delta'-\delta_b) &= \rho^2\left(\frac{\rho^2(\rho-2)}{4}\right) + (\rho-1)^2\left(\frac{\rho}{4}\left(\rho^2+2\rho-2\right)\right) \\ &\quad -2\rho(\rho-1)\left(\frac{\rho^2(\rho-2)}{4}\right) \\ &= \frac{\rho}{4}\left(-\rho^2+2\rho-2\right) \end{aligned} \tag{33}$$

The maximum value of $(-\rho^2+2\rho-2)$ is -1 when $\rho = 1$ and hence Eq. (33) is always negative for $\rho \geq 4$.

Family 2: In this case, $m+v = (\rho-1)^2$ and $m-\lambda = (\rho-2)^2$. To check the connectivity constraint of $m-\lambda$, we have $2\rho \leq \rho^2 - 4\rho + 4$ which implies that $\rho^2 - 6\rho + 4 \geq 0$ for $\rho \geq 6$. Note that $f(m+v,\rho)$ is precisely defined by Eq. (93).

Substituting Eqs. (93), (94) and (95) in Eq. (25), we obtain

$$\begin{aligned}(M-1)(\delta'-\delta_b) &= (\rho-1)^2\left(\frac{(\rho-2)(\rho^2-4\rho+6)}{4}\right)+(\rho-2)^2\left(\frac{\rho^2(\rho-2)}{4}\right)\\ &\qquad -2(\rho-1)(\rho-2)^2\left(\frac{\rho^2-2\rho+2}{4}\right)\\ &= \frac{\rho-2}{4}\Big[(\rho^2-2\rho+1)(\rho^2-4\rho+6)+\rho^2(\rho^2-4\rho+4)\\ &\qquad -2(\rho^2-3\rho+2)(\rho^2-2\rho+2)\Big]\\ &= \frac{\rho-2}{4}\Big[-\rho^2+4\rho-2\Big]<0 \quad \text{for } \rho\geq 6 \end{aligned} \tag{34}$$

Family 3: We have $m+v=(\rho+1)^2$ and $m-\lambda=\rho^2$. Therefore, $f(m-\lambda)$ is given by Eq. (92). Substituting Eqs. (92), (96) and (97) in Eq. (25), we have

$$\begin{aligned}(M-1)(\delta'-\delta_b) &= (\rho+1)^2\left(\frac{\rho(\rho^2+2\rho-2)}{4}\right)+\rho^2\left(\frac{\rho(\rho+2)(\rho+4)}{4}\right)\\ &\qquad -2\rho(\rho+1)(\rho+2)\left(\frac{\rho^2+2\rho-2}{4}\right)\\ &= \frac{\rho}{4}\Big[(\rho^2+2\rho+1)(\rho^2+2\rho-2)+\rho^2(\rho^2+6\rho+8)\\ &\qquad -2(\rho^2+3\rho+2)(\rho^2+2\rho-2)\Big]\\ &= \frac{\rho}{4}\Big[-\rho^2+2\rho+6\Big]<0 \quad \text{for } \rho\geq 4 \end{aligned} \tag{35}$$

■

Theorem 5 *Families 1 and 3 as defined in Theorem 4 are cost optimal for diameter than the 2-D balanced hypercycle* $\mathcal{G}^{\rho\ \rho}_{m\,m}$*:*

Proof.

Family 1: For $m=\rho(\rho-1)$, the diameter is given by

$$\begin{aligned}\left\lceil\frac{\lfloor\frac{m}{2}\rfloor}{\rho}\right\rceil &= \left\lceil\frac{\lfloor\frac{\rho(\rho-1)}{2}\rfloor}{\rho}\right\rceil=\left\lceil\frac{\rho(\rho-1)}{2\rho}\right\rceil\\ &= \frac{\rho}{2}\end{aligned} \tag{36}$$

For $m+v=\rho^2$, we have

$$\left\lceil\frac{\lfloor\frac{m+v}{2}\rfloor}{\rho}\right\rceil=\left\lceil\frac{\rho^2}{2\rho}\right\rceil=\frac{\rho}{2} \tag{37}$$

Similarly when $m - \lambda = (\rho - 1)^2$

$$\begin{aligned} \left\lceil \frac{\left\lfloor \frac{m-\lambda}{2} \right\rfloor}{\rho} \right\rceil &= \left\lceil \frac{\left\lfloor \frac{(\rho-1)^2}{2} \right\rfloor}{\rho} \right\rceil = \left\lceil \frac{(\rho-1)^2 - 1}{2\rho} \right\rceil \\ &= \frac{\rho - 2}{2} \end{aligned} \tag{38}$$

The difference in diameter between the non-balanced hypercycle and the balanced hypercycle can be obtained by substituting Eqs. (36), (37) and (38) in Eq. (31) as follows:

$$\begin{aligned} k' - k_b &= \frac{\rho}{2} + \frac{\rho - 2}{2} - 2\left(\frac{\rho}{2}\right) \\ &= -1 \end{aligned} \tag{39}$$

Family 3: Substituting $m = \rho(\rho + 1)$, $m + v = (\rho + 1)^2$ and $m - \lambda = \rho^2$ in Eq. (30), we obtain

$$\begin{aligned} k' - k_b &= \left\lceil \frac{\left\lfloor \frac{(\rho+1)^2}{2} \right\rfloor}{\rho} \right\rceil + \left\lceil \frac{\left\lfloor \frac{\rho^2}{2} \right\rfloor}{\rho} \right\rceil - 2\left\lceil \frac{\left\lfloor \frac{\rho(\rho+1)}{2} \right\rfloor}{\rho} \right\rceil \\ &= \left\lceil \frac{(\rho+1)^2 - 1}{2\rho} \right\rceil + \left\lceil \frac{\rho^2}{2\rho} \right\rceil - 2\left\lceil \frac{\rho(\rho+1)}{2\rho} \right\rceil \\ &= \frac{\rho + 2}{2} + \frac{\rho}{2} - 2\left(\frac{\rho + 2}{2}\right) = -1 \end{aligned} \tag{40}$$

■

Remark: If the difference $k' - k_b$ is computed for Family 2 by substituting $m = (\rho - 1)(\rho - 2)$, $m - \lambda = (\rho - 2)^2$ and $m + v = (\rho - 1)^2$ in Eq. (30), then

$$\begin{aligned} k' - k_b &= \left\lceil \frac{\left\lfloor \frac{(\rho-1)^2}{2} \right\rfloor}{\rho} \right\rceil + \left\lceil \frac{\left\lfloor \frac{(\rho-2)^2}{2} \right\rfloor}{\rho} \right\rceil - 2\left\lceil \frac{\left\lfloor \frac{(\rho-1)(\rho-2)}{2} \right\rfloor}{\rho} \right\rceil \\ &= \left\lceil \frac{(\rho-1)^2 - 1}{2\rho} \right\rceil + \left\lceil \frac{(\rho-2)^2}{2\rho} \right\rceil - 2\left\lceil \frac{(\rho-1)(\rho-2)}{2\rho} \right\rceil \\ &= \frac{\rho - 2}{2} + \frac{\rho - 2}{2} - 2\left(\frac{\rho - 2}{2}\right) \\ &= 0 \end{aligned} \tag{41}$$

Hence the non-balanced hypercycle for Family 2 is no less cost optimal for diameter than the balanced hypercycle. The exceptions stipulated by Theorem 3 shall be further discussed in Section 4.3.

Table 1 *Average Distance Minimization for 3-D hypercycle - Case 1*

Input		Theoretical Optimum						Numerical Optimum						
Nodes M	C	Node Radices*			Connectivity*			Node Radices*			Connectivity*			Average Distance
		m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	
128	5	5.09	5.09	5.09	1.67	1.67	1.67	4	4	8	1	1	2	3.0236
625	6	8.55	8.55	8.55	2	2	2	25	5	5	4	1	1	4.3269
1000	10	10	10	10	3.33	3.33	3.33	10	10	10	3	3	4	3.4034
2744	9	14	14	14	3	3	3	14	14	14	3	3	3	4.5016
27000	30	30	30	30	10	10	10	30	30	30	10	10	10	3.8001
32768	21	32	32	32	7	7	7	32	32	32	7	7	7	4.7813

Table 2 *Violations in average distance minimization for 3-D hypercycles - Case 1*

Input		Theoretical Optimum							Numerical Optimum						
Nodes M	C	Node Radices*			Connectivity*			Average Distance	Node Radices*			Connectivity*			Average Distance
		m_1	m_2	m_3	ρ_1	ρ_2	ρ_3		m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	
1728	12	12	12	12	4	4	4	3.5020	9	12	16	4	4	4	3.4325
8000	21	20	20	20	7	7	7	3.6005	16	20	25	7	7	7	3.5605
27000	33	30	30	30	11	11	11	3.6001	25	24	45	11	11	11	3.5069
46656	42	36	36	36	14	14	14	3.5000	27	32	54	14	14	14	3.4699
343000	69	70	70	70	23	23	23	3.9429	50	70	98	23	23	23	3.9155

4.3. Numerical Experiments

In the previous section, the optimal hypercycle network design for minimum average distance has been formulated as a subclass of the more general mixed discrete constrained nonlinear programming (MDCNP) problem. Several methods have been developed for solving MDCNP problems and the important ones include the *direct search methods, discrete gradient search (Steepest Descent), random discrete search, and branch and bound* type algorithms [21]. These methods require a large amount of storage, exhibit exponential increase in computational complexity as the number of design variables increases, and sometimes fail to converge to the optimal point. Recently, the Discrete Recursive Quadratic Programming (DRQP) algorithm [22] has been shown to be an effective tool for solving general MDCNP problems. An optimization package employing the DRQP has been used as one of the tools in the numerical experiments and the veracity of the results has been verified through a C program that exhaustively enumerates the various combinations in the search space Ω for small values of r. The results of discrete optimization for the various cases are presented in Tables 1 to 8 corresponding to 3-D hypercycles. The optimal point calculated theoretically is listed along with the obtained numerical value.

Remarks:

- The numerical solution, given in Table 1, agrees with the analytical solution for all cases $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i = 0$, $i = 1, \ldots, r$.

Table 3 *Violations in diameter for 3-D hypercycles - Case 1*

Input		Theoretical Optimum							Numerical Optimum						
Nodes M	C	Node Radices*			Connectivity*			Diameter	Node Radices*			Connectivity*			Diameter
		m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	k	m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	k
216	6	6	6	6	2	2	2	6	4	6	9	2	2	2	5
1728	12	12	12	12	4	4	4	6	8	9	24	4	4	4	5
8000	9	20	20	20	3	3	3	12	16	20	25	3	3	3	11
13824	27	24	24	24	9	9	9	6	18	24	32	9	9	9	5
27000	42	30	30	30	14	14	14	6	20	25	54	14	14	14	4
373248	24	72	72	72	8	8	8	15	64	72	81	8	8	8	14

Table 4 *Statistics of violations for Average distance*

Dimension of hypercycle	No. of combinations	No. of violations	Relative No. of violations	Peak relative difference $\frac{\lvert\delta_b-\delta'\rvert}{\delta_b}$
2	114856	1793	1.5611 %	3.3766 %
3	2040895	3532	0.1731 %	3.1111 %
4	53449165	7782	0.0146 %	2.6251 %
5	292607420	10781	0.0037 %	2.1437 %

- When the analytically computed points $\vec{m}$ and $\vec{\rho}$ are not integers, then a discrete optimization has to be carried in the neighbourhood of the theoretical optimum to determine the true integer values. The theoretical minimum could therefore provide a convenient starting point for the optimization.

- When $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i \neq 0$, $i = 1, \ldots, r$ and the theoretical minimum of $\vec{m}$ and $\vec{\rho}$ are integers, there exist hypercycles which are not balanced and which have smaller average distance and diameter as compared to those of the balanced hypercycle. We shall call these cases as *violation cases* with respect to Theorems 1 and 2. Some typical counter examples for average distance and diameter are listed in Tables 2 and 3. In order to quantify these violations, an exhaustive search for graphs from a given balanced point was conducted with r-D perturbations where $r \in \{2, 3, 4, 5\}$. This search covered all non-symmetric graphs upto $k = 3.0518\text{E}10$, $d = 1245$ and $\delta = 1.5259\text{E}10$. The balanced point was chosen such that $M = m^r$, $1 \leq \rho \leq \lfloor \frac{m}{2} \rfloor$ where $m \in \{2, 3, \ldots, 250\}$ for each r as stated above. The results of the numerical experiments are summarized in Tables 4 and 5 where δ' (k') refers to the average distance (diameter) of the violation points respectively. As it can be seen a small percentage of balanced hypercycles are non-optimal and this occurs when $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i \neq 0$. In addition, the following empirical observations are noteworthy.

  - The percentage of violations decreases as r increases.

Table 5 *Statistics of violations for Diameter*

Dimension of hypercycle	No. of combinations	No. of violations	Relative No. of violations	Peak relative difference $\frac{\lvert k_b - k' \rvert}{k_b}$
2	114856	3272	2.8488 %	25.00 %
3	2040895	15455	0.7573 %	33.33 %
4	53449165	90977	0.1702 %	37.50 %
5	292607420	299461	0.1023 %	40.50 %

Table 6 *Neighbourhood of balanced point for violation cases for Average distance*

Dimension	Search boundary from balanced point $\vec{m}^*$				
	$\vec{\epsilon} = [\epsilon_1 \ \ \epsilon_2 \ \ \epsilon_3 \ \ \epsilon_4 \ \ \epsilon_5]$				
r	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
2	31.58 %	46.15 %			
3	31.58 %	30.00 %	77.78 %		
4	31.58 %	30.00 %	30.00 %	87.50 %	
5	31.25 %	30.00 %	30.00 %	30.00 %	100 %

– The neighbourhood of the balanced point which contains the true optimum for δ when $X \neq 0$ is listed in Table 6. Each constant ϵ_i, $1 \leq i \leq 5$, represents the maximum relative deviation from a given balanced point $\vec{m}^*$ obtained empirically.

– A violation for average distance does not necessarily imply a violation in diameter and vice versa.

– For the diameter, the number of violations are larger compared to that of the average distance for any given r and this can be attributed to the fact that the bounds of k used in the analytical derivation are less rigid. However, no violations were observed when $\lfloor \frac{m_i}{2} \rfloor \bmod \rho_i = 0$, $i = 1, \ldots, r$.

5. Other Optimization cases

In this section, we present some analytical results for the average distance/diameter corresponding to Cases 2 and 3 and these are direct consequences of Lemmas 2 and 5.

Table 7 *Average Distance Minimization for a 3-D hypercycle - Case 2*

Input Parameters				Theoretical Optimum			Numerical Optimum			
Total Nodes	Connectivity			Node Radices*			Node Radices*			Average
M	ρ_1	ρ_2	ρ_3	m_1	m_2	m_3	m_1	m_2	m_3	Distance
81	1	2	1	3.43	6.87	3.43	3	9	3	2.7000
100	2	1	2	5.85	2.92	5.85	5	4	5	2.6263
256	4	4	2	8	8	4	8	8	4	2.5098
256	4	2	2	10.08	5.04	5.04	8	4	8	2.8862
512	4	2	1	16	8	4	16	8	4	3.6321
1024	1	1	16	4	4	16	4	4	16	3.4721
1728	3	2	1	19.81	13.21	6.60	18	16	6	5.5866
4000	1	5	3	6.44	32.18	19.31	8	25	20	5.6814
10000	8	5	4	31.75	19.84	15.87	25	20	20	4.3304
16000	4	8	3	22.01	44.03	16.51	25	40	16	5.3328

5.1. Case 2

Theorem 6 *The hypercycle $\mathcal{G}^{\vec{\rho}}_{\vec{m}}$ is cost optimal for the upper and lower bounds of the average distance and diameter when $\rho_i = c_i$ is an integer such that*

$$m_i = \sqrt[r]{\frac{M c_i^{r-1}}{\prod_{j=1, j\neq i}^{r} c_j}}, \qquad i = 1, \ldots, r \tag{42}$$

where the dimension r, the total number of nodes M and the connectivities c_i are given integer constants.

Proof. From Eq. (11), we have

$$m_t = m_1 \left(\frac{\rho_t}{\rho_1}\right) = \left(\frac{m_1}{c_1}\right) c_t, \qquad 2 \leq t \leq r$$

Then

$$\prod_{t=2}^{r} m_t = \left(\frac{m_1}{c_1}\right)^{r-1} \prod_{t=2}^{r} c_t$$

$$\frac{M}{m_1} = \left(\frac{m_1}{c_1}\right)^{r-1} \prod_{t=2}^{r} c_t$$

$$\Rightarrow m_1 = \sqrt[r]{\frac{M c_1^{r-1}}{\prod_{t=2}^{r} c_t}}$$

$$\Rightarrow m_t = \sqrt[r]{\frac{M c_t^{r-1}}{\prod_{j=1, j\neq t}^{r} c_j}}$$

The results obtained from numerical optimization are tabulated in Table 7. ■

Table 8 *Average Distance Minimization for a 3-D hypercycle - Case 3*

Input Parameters					Theoretical Optimum			Numerical Optimum			
Nodes	Connect.	Node Radices			Connectivity*			Connectivity*			Average
M	Sum	m_1	m_2	m_3	ρ_1	ρ_2	ρ_3	ρ_1	ρ_2	ρ_3	Distance
16	4	2	2	4	1.17	1.17	1.66	1	1	2	1.8667
64	6	4	4	4	2	2	2	2	2	2	2.2857
256	7	8	8	4	2.59	2.59	1.82	3	2	2	3.0117
1000	8	10	20	5	2.56	3.62	1.81	2	4	2	3.9540
1000	7	10	10	10	2.33	2.33	2.33	3	2	2	4.2042
5000	15	25	20	10	5.94	5.31	3.75	6	5	4	3.8408
12000	40	50	20	12	18.85	11.92	9.23	24	10	6	2.8669
17576	14	24	24	24	4.67	4.67	4.66	5	5	4	5.1254
46656	27	36	36	36	9	9	9	9	9	9	4.3334
64000	30	40	40	40	10	10	10	10	10	10	4.3501

5.2. Case 3

Theorem 7 *The hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ is cost optimal for the upper and lower bounds of the average distance and diameter when $m_i = c_i$ such that*

$$\rho_i = \frac{C\sqrt{c_i}}{\sqrt{c_1}+\sqrt{c_2}+\ldots+\sqrt{c_r}}, \qquad i = 1,\ldots,r \tag{43}$$

is an integer where r, C and c_i are given integer constants.

Proof. Rewriting Eq. (13), we obtain

$$\rho_t = \rho_1\sqrt{\frac{m_t}{m_1}} = \rho_1\sqrt{\frac{c_t}{c_1}}, \qquad 2 \le t \le r$$

Then

$$\begin{aligned}
\sum_{t=2}^{r}\rho_t &= \rho_1\left(\frac{\sqrt{c_2}+\ldots+\sqrt{c_r}}{\sqrt{c_1}}\right)\\
C-\rho_1 &= \rho_1\left(\frac{\sqrt{c_2}+\ldots+\sqrt{c_r}}{\sqrt{c_1}}\right)\\
\Rightarrow \rho_1 &= \frac{C\sqrt{c_1}}{\sqrt{c_1}+\sqrt{c_2}+\ldots+\sqrt{c_r}}\\
\Rightarrow \rho_t &= \frac{C\sqrt{c_t}}{\sqrt{c_1}+\sqrt{c_2}+\ldots+\sqrt{c_r}}
\end{aligned} \tag{44}$$

■

Table 8 lists the results of numerical optimization for Case 3.

5.3. Optimum Dimension

The discussion in the previous section intuitively directs us to the next question, whether it is possible to determine the optimal dimension r that will minimize the average distance δ. However, this problem is mathematically intractable owing to the variation in the node radix and connectivity vectors with the dimension r. For specific cases though, it is possible to obtain an analytical solution. For instance, consider the case of a balanced hypercycle with $m_i = m = \sqrt[r]{M}$ and $\rho_i = \rho$, $i = 1, \ldots, r$, where $M = p^a$, $m = p^b$ with a, b being integers greater than 0 and p being a given prime number. Let M and ρ be given constants with $\lfloor \frac{m}{2} \rfloor \bmod \rho = 0$. It follows immediately that $r = \log_m M = a/b$ where r is an integer. Consider the following cases.

Case 1. $p = 2$, $\rho = 1$

Then, from Eqs. (21) and (90) we obtain

$$\begin{aligned} \delta &= \frac{r m^{r-1} f(m,\rho)}{M-1} \\ &= \frac{rM(m+2(\rho-1))}{4\rho(M-1)} \\ &= \left(\frac{a}{b}\right)\left(\frac{2^a 2^b}{4(2^a-1)}\right) \\ &= c\left(\frac{2^b}{b}\right) \end{aligned} \tag{45}$$

where $c = \frac{a 2^a}{4(2^a-1)}$ is a constant.

As can be seen, Eq (45) is monotonically increasing for $b > 2$ since

$$c\left(\frac{2^b}{b} - \frac{2^{b+1}}{b+1}\right) = c\left(\frac{2^b(1-b)}{b(b+1)}\right) < 0, \quad \forall b > 2 \tag{46}$$

Hence the minimum[†] is given when $b^* = 1$ or 2. Hence the optimum dimension r can be taken as $r = \log_2 M$.

Case 2. $p > 2$, $\rho = 1$

In this case, p is odd and substituting Eq. (88) in Eq. (21), we have

$$\begin{aligned} \delta &= \frac{rM(m-1)(m-1+2\rho))}{4\rho m(M-1)} \\ &= \left(\frac{a}{b}\right)\left(\frac{p^a(p^{2b}-1)}{4p^b(p^a-1)}\right) \quad \text{since } \rho = 1 \\ &= c\left(\frac{p^{2b}-1}{bp^b}\right) \end{aligned} \tag{47}$$

[†]In essence, the largest number of dimensions minimizes the average distance and since $\rho = 1$, this corresponds to the largest degree graph. Also observe that $b^* = 2$ and $b^* = 1$ corresponds to the same graph since $\mathcal{G}_4^1$ is isomorphic to $\mathcal{G}_{2\ 2}^{1\ 1}$

where $c = \frac{ap^a}{4(p^a-1)}$ is a constant. Then

$$\frac{c(p^{2b}-1)}{bp^b} - \frac{c(p^{2(b+1)}-1)}{(b+1)p^{b+1}} = \frac{c\left((b+1)p(p^{2b}-1) - b(p^{2(b+1)}-1)\right)}{b(b+1)p^{b+1}} \quad (48)$$

Simplifying the numerator, we obtain

$$c\left((b+1)p(p^{2b}-1) - b(p^{2(b+1)}-1)\right) = c\left(p^{2b+1}(b+1-bp) + (b-(b+1)p)\right) < 0$$

$\forall b \geq 1, p > 2$. Thus the minimum of Eq (47) is achieved when $b^* = 1$ and $r = \log_p M$ gives the optimum dimension.

The above discussions can be summarized as follows:

Lemma 16 *When M is an integral power of p where p is prime and $\rho = 1$, a hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ with r dimensions minimizes the average distance such that $r = \log_p M$.*

6. Minimization of Degree*Diameter product

The degree*diameter product is of significant importance in designing a parallel network system. The parameter χ measures the dependence of the diameter on the degree and hence the smaller the degree, the larger is the diameter and vice versa. The degree*diameter product is given by

$$\chi = d \times k = \sum_{i=1}^{r} d_i \times \sum_{i=1}^{r} k_i \quad (49)$$

Due to the increased variability of the node radix and connectivity vectors, the optimization of χ is generally difficult. As an important special case, let us determine the optimum dimension r that will minimize χ corresponding to a balanced hypercycle. It has been shown in Theorems 1 and 2 that the balanced hypercycle has a cost optimal diameter when $m_i = m = \sqrt[r]{M}$ and $\rho_i = \rho = C/r$ and $\lfloor \frac{m}{2} \rfloor \bmod \rho = 0, \forall i \in \{1, 2, \ldots, r\}$, where C is a given constant. Let $M = 2^a$ and $\rho = 2^l$ be given integer constants where $l \geq 0$ and $a \geq 1$. Furthermore, let $m = 2^b$ where b is an integer and $b \geq l+1$ (since $m \geq 2\rho$). Obviously $\lfloor \frac{m}{2} \rfloor \bmod \rho = 2^{b-1} \bmod 2^l = 0$. So, if $r = a/b$ is an integer then $m = \sqrt[r]{M}$. To compute the optimum value of χ and hence r, we consider the following cases.

Case 1: $2\rho < m$

Here $b > l+1$ and it implies that the hypercycle is not fully connected in each dimension. Then

$$\chi = r \times 2\rho \times r \left\lceil \frac{m}{2\rho} \right\rceil = r^2 m = a^2 \left(\frac{2^b}{b^2} \right) \quad (50)$$

To minimize χ, we treat b as the design parameter. By setting the derivative of χ w.r.t. b to zero, we have $\frac{\partial \chi}{\partial b} = \frac{a^2 2^b}{b^3}(b \log 2 - 2) = 0$ which leads to the real solution $b' = 2/\log 2 = 2.8854$. Since b is an integer, the discrete optimum of b can be either 2

or 3. Substituting $b = 2, 3$ in Eq. 50 and comparing, we obtain $\frac{4a^2}{4} > \frac{8a^2}{9}$. Hence $b^* = 3$ is the integer minimum and consequently $m^* = 8$. Since $b^* > l+1$, it implies that $l < b^* - 1$ and hence $l \in \{0, 1\}$. Therefore $\rho = \{1, 2\}$.

Note that the second order derivative of χ is given by

$$\frac{\partial^2 \chi}{\partial b^2} = \frac{a^2[b2^b \log 2\,(b \log 2 - 4) + 6(2^b)]}{b^4}$$

which is positive for all integer values of $b \geq 1$. Thus, when $\rho = 1$, $b^* = 3$ is indeed a minimum of χ and if $r = a/b$ is an integer, it implies that a should be a multiple of 3. Hence the following lemma.

Lemma 17 *A hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ with r^* dimensions has a cost optimal degree*diameter product such that $r^* = \log_8 M$ if M is a power of 8 and $\rho = 1$.*

Remark: It should be noted that when $\rho = 1$, the hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ degenerates to a nearest neighbour hypercube. Bhuyan and Agrawal [17] have conjectured that a nearest neighbour hypercube with $r = \lceil \log_8 M \rceil$ dimensions minimizes χ when M is a power of 2 and this observation is drawn primarily from numerical experiments. In the above derivation, we have provided the mathematical basis for Bhuyan and Agrawal's conjecture as a special case in the optimization of χ for hypercycles. The aforementioned conjecture has been subsequently proved by Chen *et al.* and the interested reader is referred to [19] for more details.

Case 2: $m = 2\rho$

The degree*diameter product can be expressed as

$$\chi = r(2\rho - 1) \times r\left(\frac{m}{2\rho}\right) = r^2(m-1) = \frac{a^2(2^b - 1)}{b^2} \tag{51}$$

Setting the first order derivative of Eq. 51 to zero, we obtain $\frac{\partial \chi}{\partial b} = \frac{a^2}{b^3}\left(2^b b \log 2 - 2(2^b - 1)\right) = 0$ which has the real solution $b' = 2.2991$. Since b is an integer, the discrete optimum b can be either 2 or 3. Substituting $b = 2, 3$ in Eq. 51 and comparing, we obtain $\frac{3a^2}{4} < \frac{7a^2}{9}$. Hence $b^* = 2$ is the integer minimum and consequently $m^* = 4$ and $\rho = 2$. Since $b^* = l + 1$, it implies $l = 1$ and hence $\rho = 2$.

The second order derivative of eqn. 51 is given by

$$\frac{\partial^2 \chi}{\partial b^2} = \frac{a^2[b2^b \log 2(b \log 2 - 4) + 6(2^b - 1)]}{b^4} \tag{52}$$

which is also positive for all integer values of $b \geq 1$ and therefore $b^* = 2$ is the minimum.

Remark: For $\rho = 2$, we have obtained two optimum values of b^*, namely $b^* = 3$ and $b^* = 2$ corresponding to Cases 1 and 2 respectively. By substituting the respective values in Eqs. (50) and (51), the global minimum for $\rho = 2$ can be ascertained. Indeed, observe that $\frac{8a^2}{9} > \frac{3a^2}{4}$ and hence $b^* = 2$ is the optimum solution for $\rho = 2$. This implies that a should be a multiple of 2 if $r = a/b$ is an integer and hence we have the following theorem.

Theorem 8 *A balanced hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ where $\vec{m} = [4\ 4 \ldots 4\ 4]$ and $\vec{\rho} = [2\ 2 \ldots 2\ 2]$ with r^* dimensions has a cost optimal degree*diameter product such that $r^* = \log_4 M$ if M is an even power of 2 and $\rho = 2$.*

Proof. Let the node radices of the optimal hypercycle be represented as $m_t = 4$, $\forall t$ such that $1 \leq t \leq r^*$ and $\rho_t = 2$ and furthermore let $M = 2^a$. Then $r^* = a/2$ and from Eq. (51), we have

$$\chi^* = \frac{3a^2}{4} \tag{53}$$

The optimality of Eq. (53) can be established by comparing χ^* for all valid hypercycle topologies satisfying the constraints $\rho_t = 2$, $m_t \geq 2\rho_t$, $\forall t$ $1 \leq t \leq r$, such that $\nabla = \chi - \chi^* \geq 0$. Note that r is bounded from above by r^* since otherwise if $r > r^*$, the connectivity constraint will be violated as $\exists m_i$ such that $m_i \leq 2\rho_i$, $1 \leq i \leq r$. However, before proceeding to the actual proof, we state the following Lemmas which are important for subsequent discussion.

Lemma 18 *For given integers x and y such that $x, y \geq 3$,*

$$2^x + 2^y - \frac{3}{2}xy \geq \frac{5}{2} \tag{54}$$

Proof. The lower bound of Eq. (54) can be obtained by substituting $x = y = 3$ and the difference between successive function values for unit perturbations in x and y can be computed as

$$\begin{aligned} \epsilon &= 2^{x+1} + 2^{y+1} - \frac{3}{2}(x+1)(y+1) - \left(2^x + 2^y - \frac{3}{2}xy\right) \\ &= 2^x + 2^y - \frac{3}{2}(x+y) - \frac{3}{2} > 0 \quad \text{for } x, y \geq 3 \end{aligned}$$

■

Lemma 19 *For given integers n, b_i where $1 \leq i \leq n$ and $b_i \geq 3$,*

$$(n-1)\sum_{i=1}^{n} 2^{b_i} - \frac{3}{2}\sum_{t=1}^{n-1}\sum_{j=t+1}^{n} b_t b_j \geq \frac{5n(n-1)}{4} \tag{55}$$

Proof. We can express Eq. (55) as

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(2^{b_i} + 2^{b_j} - \frac{3}{2}b_i b_j\right) \geq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{5}{2} = \frac{5n(n-1)}{4} \quad \text{(From Lemma 18)} \tag{56}$$

■

Lemma 20 *For given integers x and y such that $x \geq 2$ and $y \geq 3$,*

$$2^{x+1} + \frac{5}{8}2^y - \frac{3}{2}xy \geq 4 \tag{57}$$

Proof. As in Lemma 18. ■

Lemma 21 *For given integers n, $x \geq 3$ and $y \geq 2$,*

$$2^{x-2} - x \geq -1 \tag{58}$$

$$2^x - x^2 \geq -1 \tag{59}$$

$$\frac{7}{8}2^x - \frac{3}{4}x^2 \geq \frac{1}{4} \tag{60}$$

$$2^x - \frac{3}{4}x^2 \geq \frac{5}{4} \tag{61}$$

$$\frac{1}{8}\sum_{i=1}^{n} 2^x \geq n \tag{62}$$

$$\frac{1}{4}\sum_{i=1}^{n} 2^x \geq 2n \tag{63}$$

Proof. By elementary mathematical analysis similar to Lemma 18. ■

We can now evaluate the cost function ∇ for all valid topologies such that $M = \prod_{t=1}^{r} m_t$ where $m_t \geq 4$ and $\rho_t = 2$. Let s be an integer such that $0 \leq s \leq r$ such that $m_j = 4$, $\forall j$ such that $1 \leq j \leq s$ and $m_i = 2^{b_i}$ where $b_i \geq 3$, $\forall i$ such that $(s+1) \leq i \leq r$. It follows immediately that $a = 2s + \sum_{i=s+1}^{r} b_i$. The degree of these hypercycles is given by $d = 3s + 4(r-s)$ from Eq. (1) while the diameter can be calculated from Eq. (2) to obtain $k = s + \frac{1}{4}\sum_{i=s+1}^{r} 2^{b_i}$. Therefore

$$\begin{aligned} \chi &= (3s + 4(r-s)) \times \left(s + \frac{1}{4}\sum_{i=s+1}^{r} 2^{b_i} \right) \\ &= (4r - s)\left(s + \sum_{i=s+1}^{r} 2^{b_i - 2} \right) \end{aligned} \tag{64}$$

Substituting $a = 2s + \sum_{i=s+1}^{r} b_i$ in Eq. (53), we obtain

$$\begin{aligned} \chi^* &= \frac{3}{4}\left(4s^2 + 4s\sum_{i=s+1}^{r} b_i + \left(\sum_{i=s+1}^{r} b_i \right)^2 \right) \\ &= 3s^2 + 3s\sum_{i=s+1}^{r} b_i + \frac{3}{4}\left(\sum_{i=s+1}^{r} b_i \right)^2 \end{aligned} \tag{65}$$

Subtracting Eq. (65) from Eq. (64), we have

$$\begin{aligned} \nabla &= 4rs - (4r - s)\sum_{i=s+1}^{r} 2^{b_i - 2} - 4s^2 - 3s\sum_{i=s+1}^{r} b_i - \frac{3}{4}\left(\sum_{i=s+1}^{r} b_i \right)^2 \\ &= 4rs - 4s^2 + 3s\sum_{i=s+1}^{r} 2^{b_i - 2} - b_i + 4(r-s)\sum_{i=s+1}^{r} 2^{b_i - 2} - \frac{3}{4}\left(\sum_{i=s+1}^{r} b_i \right)^2 \end{aligned}$$

Substituting Eq. (58) in the above, we have

$$\begin{aligned}\nabla &\geq 4rs - 4s^2 - 3s(r-s) + 4(r-s)\sum_{i=s+1}^{r} 2^{b_i-2} - \frac{3}{4}\left(\sum_{i=s+1}^{r} b_i\right)^2 \\ &= s(r-s) + (r-s)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{4}\left(\sum_{i=s+1}^{r} b_i\right)^2 \qquad (66)\end{aligned}$$

Subcase 1: $s = r$

In this case, $\nabla = 0$ and the case of a balanced hypercycle is obtained with $m_t = 4$, for $t = 1, \ldots, r$ and $\rho_t = 2$.

Subcase 2: $s = r - 1$

Simplifying Eq. (66), we obtain

$$\begin{aligned}\nabla &\geq s(r-s) + (r-s)2^{b_r} - \frac{3}{4}b_r^2 \\ &= (r-1) + 2^{b_r} - \frac{3}{4}b_r^2 \qquad \text{(Substituting } s = r-1) \qquad (67)\\ &\geq (r-1) + \frac{5}{4} \qquad \text{(From Eq. (61))} \\ &= r + \frac{1}{4} > 0\end{aligned}$$

Subcase 3: $s \leq r - 2$

Expanding Eq. (66), we obtain

$$\begin{aligned}\nabla &\geq s(r-s) + (r-s)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{4}\sum_{i=s+1}^{r} b_i^2 - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j \\ &= s(r-s) + \frac{3}{4}\sum_{i=s+1}^{r} 2^{b_i} - b_i^2 + \left(r-s-\frac{3}{4}\right)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j \\ &= s(r-s) + \frac{3}{4}\sum_{i=s+1}^{r} 2^{b_i} - b_i^2 + \frac{1}{4}\sum_{i=s+1}^{r} 2^{b_i} + (r-s-1)\sum_{i=s+1}^{r} 2^{b_i} \\ &\qquad - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j \\ &\geq s(r-s) - \frac{3}{4}(r-s) + 2(r-s) \qquad \text{(From Eqs. (59) and (63))} \\ &\qquad + (r-s-1)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j \\ &\geq s(r-s) - \frac{3}{4}(r-s) + 2(r-s) + \frac{5(r-s)(r-s-1)}{4} \qquad \text{(From Lemma 19)} \\ &= (r-s)\left(s + \frac{5}{4} + \frac{5(r-s-1)}{4}\right)\end{aligned}$$

$$\geq \quad 2(s+\frac{5}{2}) > 0 \quad \text{(Substituting the lower bound } r-s=2 \text{ in the previous eqn.)}$$

■

The preceding discussion naturally leads us to the question what is the optimal configuration when M is an odd power of two given that $\rho = 2$. Under this condition, it can be observed that $r = a/2$ is not an integer since $M = 2^a$ is odd. Intuitively, we can consider two possible solutions.

1. $r = \lceil \log_4 M \rceil$. Observe that there exists an m_i such that $m_i = 2$, $1 \leq i \leq r$ which would violate the connectivity constraint since $m_i < 2\rho$ for $\rho = 2$ and therefore we can preclude this solution.
2. $r = \lfloor \log_4 M \rfloor$. In this case, we have $m_t \geq 4$ for $t = 1, \ldots, r$ and it is obvious that the connectivity constraints are satisfied in all dimensions for $\rho = 2$.

We shall now state and prove the optimality of the second assumption as follows.

Theorem 9 *A hypercycle $\mathcal{G}_{\vec{m}}^{\vec{\rho}}$ with r^* dimensions has a cost optimal degree*diameter product such that $r^* = \lfloor \log_4 M \rfloor$, $\vec{m} = [4\ 4 \ldots 4\ 8\ 4 \ldots 4\ 4]$ and $\vec{\rho} = [2\ 2 \ldots\ 2\ 2\ 2 \ldots 2\ 2]$ if M is an odd power of 2 and $\rho = 2$.*

Proof. Since $M = 2^a$, we have $r^* = \lfloor \frac{a}{2} \rfloor = \frac{a-1}{2}$. The node radix of the optimal hypercycle can be represented as $m_t = 4$, $1 \leq t \leq r^*$, $t \neq i$, $1 \leq i \leq r^*$ and $m_i = M/4^{r^*-1} = 2^{a-2(r^*-1)} = 8$ with $\rho = 2$. The degree of the network is then given by $3(r^*-1)+4$ while its diameter is $r^*-1+2 = r^*+1$. Hence the degree*diameter product is given by

$$\begin{aligned} \chi^* &= (3(r^*-1)+4)(r^*+1) = (3r^*+1)(r+1) \\ &= \frac{3a^2+2a-1}{4} \end{aligned} \tag{68}$$

As before, the optimality of Eq. (68) can be established by comparing χ^* with all valid hypercycle topologies satisfying the constraints $\rho_t = 2$, $m_t \geq 2\rho_t$, $\forall t$, $1 \leq t \leq r$ such that $\nabla = \chi - \chi^* \geq 0$ where $r \leq r^*$.

Cost of Hypercycle topologies such that $M = \prod_{t=1}^{r} m_t$ where $m_t \geq 4$ and $\rho_t = 2$.

Let s be an integer such that $0 \leq s \leq r-1$ and $m_j = 4$, $1 \leq j \leq s$ with $m_i \geq 2^{b_i}$ where $b_i \geq 3$ and $(s+1) \leq i \leq r$. Observe that if $s = r$, then $\prod_{j=1}^{r} m_j = 4^r < M$ since $r \leq r^*$ and $4^{r^*} = M/2$. The degree of these topologies is $3s+4(r-s)$ while the diameter is equal to $s + \frac{1}{4}\sum_{i=s+1}^{r} 2^{b_i}$ as obtained previously. Hence χ is given by Eq. (64).

Substituting the value of $a = 2s + \sum_{i=s+1}^{r} b_i$ in Eq. (68), we obtain

$$\chi^* = \frac{3}{4}\left(4s^2 + 4s\sum_{i=s+1}^{r} b_i + \left(\sum_{i=s+1}^{r} b_i\right)^2\right) + \frac{1}{4}\left(4s + 2\sum_{i=s+1}^{r} b_i\right) - \frac{1}{4}$$

$$= 3s^2 + s + \left(3s + \frac{1}{2}\right) \sum_{i=s+1}^{r} b_i + \frac{3}{4} \left(\sum_{i=s+1}^{r} b_i \right)^2 - \frac{1}{4} \tag{69}$$

Subtracting Eq. (69) from Eq. (64), we have

$$\begin{aligned} \nabla &= 4rs - 4s^2 - s + (4r - s) \sum_{i=s+1}^{r} 2^{b_i - 2} - \left(3s + \frac{1}{2}\right) \sum_{i=s+1}^{r} b_i - \frac{3}{4} \left(\sum_{i=s+1}^{r} b_i \right)^2 + \frac{1}{4} \\ &= 4rs - 4s^2 - s + \left(3s + \frac{1}{2}\right) \sum_{i=s+1}^{r} 2^{b_i - 2} - b_i + \left(4(r - s) - \frac{1}{2}\right) \sum_{i=s+1}^{r} 2^{b_i - 2} \\ &\quad - \frac{3}{4} \left(\sum_{i=s+1}^{r} b_i \right)^2 + \frac{1}{4} \end{aligned}$$

Substituting Eq. (58) in the above, we have

$$\begin{aligned} \nabla &\geq 4rs - 4s^2 - s - \left(3s + \frac{1}{2}\right)(r - s) + \left(4(r - s) - \frac{1}{2}\right) \sum_{i=s+1}^{r} 2^{b_i - 2} - \frac{3}{4} \left(\sum_{i=s+1}^{r} b_i \right)^2 + \frac{1}{4} \\ &= s\left(r - s - \frac{1}{2}\right) - \frac{r}{2} + \left(r - s - \frac{1}{8}\right) \sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{4} \left(\sum_{i=s+1}^{r} b_i \right)^2 + \frac{1}{4} \end{aligned} \tag{70}$$

Subcase 1: $s = r - 1$

Substituting $s = r - 1$ in Eq. (70), we obtain

$$\begin{aligned} \nabla &\geq \frac{r - 1}{2} - \frac{r}{2} + \frac{7}{8} 2^{b_r} - \frac{3}{4} b_r^2 + \frac{1}{4} \qquad (71) \\ &\geq -\frac{1}{2} + \frac{1}{4} + \frac{1}{4} \qquad \text{(From Eq. (60))} \\ &= 0 \end{aligned}$$

Subcase 2: $s \leq r - 2$

Expanding Eq. (70), we obtain

$$\begin{aligned} \nabla &\geq s\left(r - s - \frac{1}{2}\right) - \frac{r}{2} + \left(r - s - \frac{1}{8}\right) \sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{4} \sum_{i=s+1}^{r} b_i^2 - \frac{3}{2} \sum_{t=s+1}^{r-1} \sum_{j=t+1}^{r} b_t b_j + \frac{1}{4} \\ &= s\left(r - s - \frac{1}{2}\right) - \frac{r}{2} + \frac{3}{4} \sum_{i=s+1}^{r} 2^{b_i} - b_i^2 + \left(r - s - \frac{7}{8}\right) \sum_{i=s+1}^{r} 2^{b_i} \\ &\quad - \frac{3}{2} \sum_{t=s+1}^{r-1} \sum_{j=t+1}^{r} b_t b_j + \frac{1}{4} \\ &= s\left(r - s - \frac{1}{2}\right) - \frac{r}{2} + \frac{3}{4} \sum_{i=s+1}^{r} 2^{b_i} - b_i^2 + \frac{1}{8} \sum_{i=s+1}^{r} 2^{b_i} \end{aligned}$$

$$
\begin{aligned}
& + (r-s-1)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j + \frac{1}{4} \\
\geq\ & s\left(r-s-\frac{1}{2}\right) - \frac{r}{2} - \frac{3}{4}(r-s) + (r-s) \quad \text{(From Eqs. (59) and (62))} \\
& + (r-s-1)\sum_{i=s+1}^{r} 2^{b_i} - \frac{3}{2}\sum_{t=s+1}^{r-1}\sum_{j=t+1}^{r} b_t b_j + \frac{1}{4} \\
\geq\ & s\left(r-s-\frac{1}{2}\right) - \frac{r}{2} + \frac{r-s}{4} + \frac{5(r-s)(r-s-1)}{4} + \frac{1}{4} \quad \text{(From Lemma 19)} \\
=\ & (r-s)\left(s+\frac{1}{4}\right) - \frac{r+s}{2} + \frac{5(r-s)(r-s-1)}{4} + \frac{1}{4} \\
=\ & \frac{(r-s)(5r-4-s)}{4} - \frac{r+s}{2} + \frac{1}{4} \\
=\ & (r-s)(r-1) + \frac{(r-s)^2}{4} - \frac{r+s}{2} + \frac{1}{4} \\
=\ & r\left(r-s-\frac{3}{2}\right) + \frac{s}{2} + \frac{(r-s)^2}{4} + \frac{1}{4} \\
\geq\ & \frac{r+s}{2} + \frac{5}{4} > 0 \qquad \text{(Substituting the lower bound } r-s=2 \text{ in the above)}
\end{aligned}
$$

■

6.1. Other Optimization Cases:

Let $M = p^a$, $m = p^b$ where p is a prime number greater than 2 and let $\lfloor \frac{m}{2} \rfloor \bmod \rho = 0$. Then χ can be rewritten as

$$\chi = r2\rho * r\left\lceil \frac{\lfloor \frac{m}{2} \rfloor}{\rho} \right\rceil = r^2(m-1) = \frac{a^2(p^b-1)}{b^2} \tag{72}$$

Setting the first order derivative of Eq. (72) to zero, we obtain $\frac{\partial \chi}{\partial b} = \frac{a^2}{b^3}\left(bp^b \log p - 2(p^b-1)\right) = 0$.

1. $p = 3$

   The analytical minimum is given by $b' = 1.4506$ and there exists two local integer optimums namely $b^* = 1$ and 2. The second order derivative given by

   $$\frac{\partial^2 \chi}{\partial b^2} = \frac{a^2[bp^b \log p\,(b \log p - 4) + 6(p^b-1)]}{b^4} \tag{73}$$

   is positive for all integer values of $b \geq 1$. Hence the following Lemma.

   Lemma 22 *A hypercycle with r^* dimensions has an optimal degree*diameter product such that*

   $$r^* = \begin{cases} \log_3 M & \text{when } M \text{ is a power of 3 and } \rho = 1 \\ \log_9 M & \text{when } M \text{ is an even power of 3 and } \rho \in \{2,4\} \end{cases}$$

2. $p = 5$
   The analytical solution for b is given by $b' = 0.9902$ and the integer minimum is given by $b^* = 1$.

   Lemma 23 *When M is a power of 5, a hypercycle with r dimensions has an optimal degree*diameter product such that $r = \log_5 M$ if $\rho = c$ where $c \in \{1, 2\}$.*

7. Network Latency and Throughput

Hypercycles can be modeled as an M/M/1 communication network [17] in which the expression for the network latency T is given as

$$\begin{aligned} T &= \frac{\delta |\mathcal{E}_m^{\rho}|}{\mu C_p (1 - \delta\gamma)} \\ &= \frac{\delta M d}{2\mu C_p (1 - \delta\gamma)} \end{aligned} \tag{74}$$

where C_p is the capacity of the network, μ is the average service rate, δ is the average distance and γ being the utilization factor. The capacity of the network is the maximum bandwidth of the network while the utilization factor γ is the number of messages handled by the network and is normalized to the capacity C_p. In the above model, it is assumed that messages are independent and uniformly distributed. Furthermore, messages are routed using the shortest path or greedy routing [20] assuming packet switching with infinite queueing.

Taking μ, C_p and M as constants and normalizing the network latency, we obtain

$$T' = \frac{\delta d}{(1 - \delta\gamma)} \tag{75}$$

The network delay increases exponentially as the utilization factor increases, and the delay asymptotically saturates when $\gamma = 1/\delta$. Similarly, the average traffic density, T_d, is given by

$$\begin{aligned} T_d &= \frac{\text{Average distance} \times \text{Total number of nodes}}{\text{Total number of edges}} \\ &= \frac{\delta . M}{\left(\frac{Md}{2}\right)} = \frac{2\delta}{d} \end{aligned} \tag{76}$$

Hence the smaller the average distance, the greater is the utilization factor and the smaller is the network delay and traffic density for a given load. A plot of the normalized network latency is illustrated in Fig. 2 for different hypercycles when $M = 256$, $d = 8$, and ρ_i being a power of two. The *throughput* T_p of the network is defined as the number of messages handled by the network per unit time, and is inversely proportional to the traffic density. Therefore

$$T_p = \frac{M}{T_d T}$$

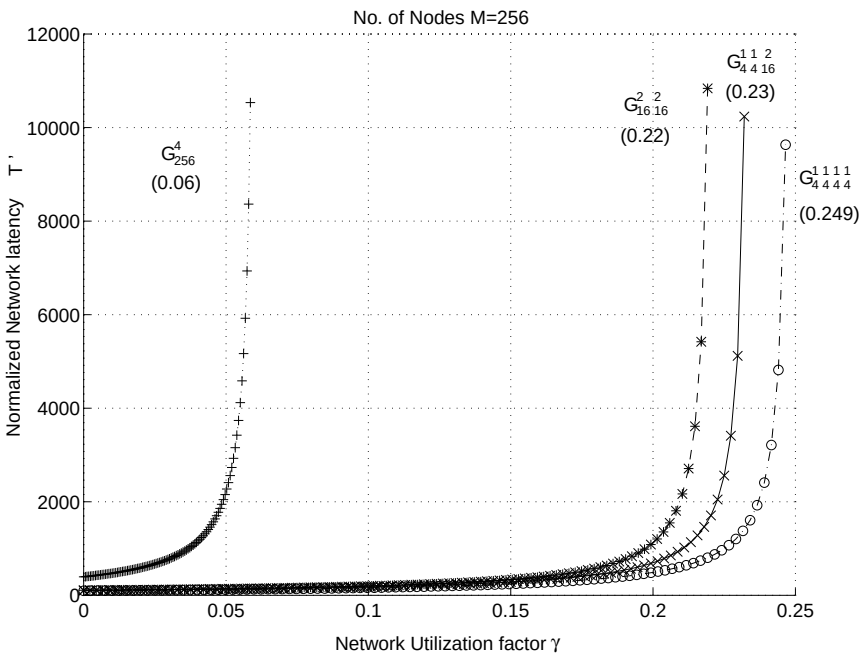

Figure 2 *Plot of normalized network latency in Hypercycles for $M = 256$ and $d = 8$. The maximum network utilization for each of the cases is given in brackets.*

$$= \frac{\left(\frac{Md}{2\delta}\right)}{T}$$
$$\propto \frac{(1-\delta\gamma)}{\delta^2}$$

Thus the normalized throughput of the network is given as

$$T'_p = \frac{(1-\delta\gamma)}{\delta^2}$$

8. Conclusion

This work has focussed on hypercycles which are a class of multidimensional, symmetric graphs that are generalizations of several interconnection networks, including hypercubes and GHCs. They exhibit wide topological flexibility, variable connectivity and higher degree of fault-tolerance. In this report, a rigorous mathematical analysis of the average distance and diameter has been attempted and it provides some insight into the optimal design of hypercycle topologies. Analytical solutions, wherever possible have been derived and this has not been addressed in detail in the open literature to the best of the author's understanding. It has been shown in this work that a balanced hypercycle is an optimal network only when $X = \lfloor \frac{m}{2} \rfloor \bmod \rho = 0$. When $X \neq 0$, the balanced hypercycle degenerates into a quasi-optimal[‡] structure and some interesting families of counter examples have been found. It is a significant result that shows that a well balanced and symmetrical network is not necessarily the true optimum for arbitrary input parameters. The exposition of this result applies to all graphs encompassed by the hypercycle including the hypercubes, k-ary hypercubes, GHC, toruses and circulant graphs. For the r-D torus, hypercubes and k-ary hypercubes, $\rho_i = 1$, while for the GHC, $\rho_i = \lfloor \frac{m_i}{2} \rfloor$, $i = 1, \ldots, r$. It immediately follows that $X = 0$ in all these cases and hence the balanced network provides an optimal structure. For cartesian products of circulant graphs (similar to hypercycles) whose connectivity $\rho_i \geq 2$, the abnormality presented here applies.

The hypercycle networks have been developed primarily for message passing concurrent computers and the technological strides in system integration and the rapidly decreasing hardware costs have heralded the arrival of VLSI communication chips. As a practical application, a 1.2 μm CMOS deadlock-free router chip [23] incorporating *prioritized dimension* routing has been designed and implemented for 4-dimensional hypercycle networks featuring up to 15 nodes per dimension with a maximum degree of 16. The measured throughput of the chip is in excess of 16 Million decisions per second.

[‡]A quasi-optimal point is a point which lies in a neighbourhood of the true optimum.

Acknowledgement

This research work has been supported by the Natural Sciences and Engineering Research Council of Canada through operating and strategic grants, by the Institute for Robotics and Intelligent Systems of the National Network of Centres of Excellence and the University of Victoria through a graduate fellowship. We also gratefully acknowledge Drs. W.-S. Lu and Z. Dong for their technical comments on the paper. Our gratitude to Mr. V. Dimakopoulos for contributing to the proof of Theorem 4.

Appendix A.

Proof of Lemma 2 The upper bound of δ can be expressed as

$$\hat{\delta} = \frac{\sum_{j=1}^{r} \hat{S}_j}{M-1}$$

$$\begin{aligned}
\hat{S}_j &= \prod_{k=1,k\neq j}^{r} m_k \sum_{\beta_j=0}^{m_j-1} \hat{D}_{\rho_j}(\beta_j, 0) \\
\hat{D}_{\rho_j}(\beta_j, 0) &= \begin{cases} \frac{\beta_j}{\rho_j} + 1 & \text{if } \beta_j \leq (m_j - \beta_j) \\ \frac{m_j - \beta_j}{\rho_j} + 1 & \text{otherwise} \end{cases}
\end{aligned}$$

We partition $\hat{\delta}$ as the sum of the distances in dimension 1 and the remaining $r-1$ dimensions of $\mathcal{G}_m^\rho$ such that

$$\begin{aligned}
\hat{\delta}.(M-1) &= \prod_{h=2}^{r} m_h \left(\sum_{i=\lceil \frac{m_1}{2} \rceil}^{m_1-1} \left(\frac{m_1 - i}{\rho_1} + 1 \right) + \sum_{i=0}^{\lceil \frac{m_1}{2} \rceil - 1} \left(\frac{i}{\rho_1} + 1 \right) \right) \\
&\quad + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(\sum_{\beta_j=\lceil \frac{m_j}{2} \rceil}^{m_j-1} \left(\frac{m_j - \beta_j}{\rho_j} + 1 \right) + \sum_{\beta_j=0}^{\lceil \frac{m_j}{2} \rceil - 1} \left(\frac{\beta_j}{\rho_j} + 1 \right) \right) \\
&= \prod_{h=2}^{r} m_h \left(m_1 + \frac{1}{\rho_1} \left(\sum_{i=\lceil \frac{m_1}{2} \rceil}^{m_1-1} m_1 - i + \sum_{i=0}^{\lceil \frac{m_1}{2} \rceil - 1} i \right) \right) \\
&\quad + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(m_j + \frac{1}{\rho_j} \left(\sum_{\beta_j=\lceil \frac{m_j}{2} \rceil}^{m_j-1} m_j - \beta_j + \sum_{\beta_j=0}^{\lceil \frac{m_j}{2} \rceil - 1} \beta_j \right) \right) \\
&= \prod_{h=2}^{r} m_h \left(m_1 + \frac{1}{\rho_1} \left\lfloor \frac{m_1^2}{4} \right\rfloor \right) + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(m_j + \frac{1}{\rho_j} \left\lfloor \frac{m_j^2}{4} \right\rfloor \right) \\
&\leq \prod_{h=2}^{r} m_h \left(m_1 + \frac{m_1^2}{4\rho_1} \right) + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(m_j + \frac{m_j^2}{4\rho_j} \right) \\
&= \prod_{h=1}^{r} m_h \left(1 + \frac{m_1}{4\rho_1} \right) + \sum_{j=2}^{r} \prod_{k=1,}^{r} m_k \left(1 + \frac{m_j}{4\rho_j} \right) \\
&= M \left(\frac{m_1}{4\rho_1} + \sum_{j=2}^{r} \frac{m_j}{4\rho_j} + r \right)
\end{aligned}$$

Similarly the lower bound of δ is given by $\check{\delta} = \frac{\sum_{j=1}^{r} \check{S}_j}{M-1}$ where

$$\begin{aligned}
\check{S}_j &= \prod_{k=1,k\neq j}^{r} m_k \sum_{\beta_j=0}^{m_j-1} \check{D}_{\rho_j}(\beta_j, 0) \\
\check{D}_{\rho_j}(\beta_j, 0) &= \begin{cases} \frac{\beta_j}{\rho_j} & \text{if } \beta_j \leq (m_j - \beta_j) \\ \frac{m_j - \beta_j}{\rho_j} & \text{otherwise} \end{cases}
\end{aligned}$$

and we obtain

$$\begin{aligned}
\check{\delta}.(M-1) &= \prod_{h=2}^{r} m_h \left(\sum_{i=\lceil \frac{m_1}{2} \rceil}^{m_1-1} \left(\frac{m_1 - i}{\rho_1} \right) + \sum_{i=0}^{\lceil \frac{m_1}{2} \rceil - 1} \left(\frac{i}{\rho_1} \right) \right) \\
&\quad + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(\sum_{\beta_j=\lceil \frac{m_j}{2} \rceil}^{m_j-1} \left(\frac{m_j - \beta_j}{\rho_j} \right) + \sum_{\beta_j=0}^{\lceil \frac{m_j}{2} \rceil - 1} \left(\frac{\beta_j}{\rho_j} \right) \right) \\
&= \prod_{h=2}^{r} m_h \left(\frac{1}{\rho_1} \left\lfloor \frac{m_1^2}{4} \right\rfloor \right) + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(\frac{1}{\rho_j} \left\lfloor \frac{m_j^2}{4} \right\rfloor \right)
\end{aligned}$$

Observe that when m_t is odd, $1 \leq t \leq r$, $\left\lfloor \frac{m_t^2}{4} \right\rfloor = \frac{m_t^2 - 1}{4}$ and when m_t is even, $\left\lfloor \frac{m_t^2}{4} \right\rfloor \geq \frac{m_t^2 - 1}{4}$. Hence

$$\begin{aligned}
\check{\delta}.(M-1) &\geq \prod_{h=2}^{r} m_h \left(\frac{m_1^2 - 1}{4\rho_1} \right) + \sum_{j=2}^{r} \prod_{k=1,k\neq j}^{r} m_k \left(\frac{m_j^2 - 1}{4\rho_j} \right) \\
&= \prod_{h=1}^{r} m_h \left(\frac{m_1}{4\rho_1} - \frac{1}{4} \right) + \sum_{j=2}^{r} \prod_{k=1,}^{r} m_k \left(\frac{m_j}{4\rho_j} - \frac{r-1}{4} \right) \\
&= M \left(\frac{m_1}{4\rho_1} + \sum_{j=2}^{r} \frac{m_j}{4\rho_j} - \frac{r}{4} \right)
\end{aligned}$$

■

Appendix B.

Proof of Lemma 6 Rewriting Eq. (6), we obtain

$$f(m, \rho) = \sum_{i=0}^{\lceil \frac{m}{2} \rceil - 1} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{i=\lceil \frac{m}{2} \rceil}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil$$

Consider the following cases:

1. m **- odd**

$$\begin{aligned} f(m,\rho) &= \sum_{i=1}^{\frac{m-1}{2}} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{i=\frac{m+1}{2}}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil \\ &= \sum_{i=1}^{\frac{m-1}{2}} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{j=\frac{m-1}{2}}^{1} \left\lceil \frac{j}{\rho} \right\rceil \quad \text{(substituting } i = m-j \text{ in the II term of the above)} \\ &= 2 \sum_{i=1}^{\frac{m-1}{2}} \left\lceil \frac{i}{\rho} \right\rceil \end{aligned}$$

2. m **even**

$$\begin{aligned} f(m,\rho) &= \sum_{i=0}^{\frac{m}{2}-1} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{i=\frac{m}{2}}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil \\ &= \sum_{i=1}^{\frac{m}{2}-1} \left\lceil \frac{i}{\rho} \right\rceil + \left\lceil \frac{m}{2\rho} \right\rceil + \sum_{i=\frac{m+2}{2}}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil \\ &= \left(\sum_{i=1}^{\frac{m}{2}-1} \left\lceil \frac{i}{\rho} \right\rceil + \left\lceil \frac{m}{2\rho} \right\rceil \right) + \left(\sum_{i=\frac{m+2}{2}}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil + \left\lceil \frac{m}{2\rho} \right\rceil \right) - \left\lceil \frac{m}{2\rho} \right\rceil \\ &= \sum_{i=1}^{\frac{m}{2}} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{i=\frac{m}{2}}^{m-1} \left\lceil \frac{m-i}{\rho} \right\rceil - \left\lceil \frac{m}{2\rho} \right\rceil \\ &= \sum_{i=1}^{\frac{m}{2}} \left\lceil \frac{i}{\rho} \right\rceil + \sum_{j=\frac{m}{2}}^{1} \left\lceil \frac{j}{\rho} \right\rceil - \left\lceil \frac{m}{2\rho} \right\rceil \quad \text{(substituting } i = m-j \text{ in the II term of the above)} \\ &= 2 \sum_{i=1}^{\frac{m}{2}} \left\lceil \frac{i}{\rho} \right\rceil - \left\lceil \frac{m}{2\rho} \right\rceil \end{aligned}$$

■

Proof of Lemma 7:
In this section, we first derive a closed formula for computing $\sum_{i=1}^{n} \left\lceil \frac{i}{\rho} \right\rceil$. For given integers ρ and $i \geq 1$, we write

$$i = \rho x_i + r_i \tag{77}$$

where x_i and r_i are integers such that $x_i \geq 0$ and $1 \leq r_i \leq \rho$. Note that such a representation of i is unique. It follows that

$$\left\lceil \frac{i}{\rho} \right\rceil = x_i + 1 \tag{78}$$

$$\left\lfloor \frac{i}{\rho} \right\rfloor = \begin{cases} x_i & \text{if } 1 \le r_i \le \rho - 1 \\ x_i + 1 & \text{if } r_i = \rho \end{cases} \tag{79}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{n} \left\lfloor \frac{i}{\rho} \right\rfloor &= \sum_{i=1}^{n} (x_i + 1) \\ &= n + \sum_{i=1}^{n} \frac{i - r_i}{\rho} \\ &= n + \frac{n(n+1)}{2\rho} - \frac{1}{\rho} \sum_{i=1}^{n} r_i \end{aligned} \tag{80}$$

In order to evaluate the term $\frac{1}{\rho} \sum_{i=1}^{n} r_i$, we consider the following cases:

Case 1: n/ρ is an integer

It can be observed that for $1 \le i \le \rho$, we have $x_i = 0$ and $r_i = i$. Hence $\sum_{i=1}^{\rho} r_i = \frac{\rho(\rho+1)}{2}$. Similarly, for $(\rho + 1) \le i \le 2\rho$, we have $x_i = 1$ and $r_i = i$. So $\sum_{i=\rho+1}^{2\rho} r_i = \frac{\rho(\rho+1)}{2}$. In general, we obtain $\sum_{i=\rho(j-1)+1}^{\rho j} r_i = \frac{\rho(\rho+1)}{2}$ for $j \in \left\{1, 2, \ldots, \frac{n}{\rho}\right\}$. Therefore

$$\begin{aligned} \frac{1}{\rho} \sum_{i=1}^{n} r_i &= \frac{1}{\rho} \underbrace{\left[\sum_{i=1}^{\rho} r_i + \sum_{i=\rho+1}^{2\rho} r_i + \ldots + \sum_{i=n-2\rho+1}^{n-\rho} r_i + \sum_{i=n-\rho+1}^{n} r_i \right]}_{n/\rho \quad \text{terms}} \\ &= \left(\frac{n}{\rho}\right) \frac{(\rho+1)}{2} \\ &= \left\lfloor \frac{n}{\rho} \right\rfloor \frac{(\rho+1)}{2} \end{aligned} \tag{81}$$

Case 2: n/ρ is not an integer

$$\begin{aligned} \frac{1}{\rho} \sum_{i=1}^{n} r_i &= \frac{1}{\rho} \sum_{i=1}^{\left\lfloor \frac{n}{\rho} \right\rfloor \rho} r_i + \frac{1}{\rho} \sum_{i=\left\lfloor \frac{n}{\rho} \right\rfloor \rho + 1}^{n} r_i \\ &= \left\lfloor \frac{n}{\rho} \right\rfloor \frac{(\rho+1)}{2} + \frac{1}{\rho} \sum_{i=\left\lfloor \frac{n}{\rho} \right\rfloor \rho + 1}^{n} r_i \end{aligned} \tag{82}$$

Let $i = \rho \left\lfloor \frac{n}{\rho} \right\rfloor + j$ where $j \in \left\{1, 2, \ldots, n - \left\lfloor \frac{n}{\rho} \right\rfloor \rho\right\}$. Since $n - \left\lfloor \frac{n}{\rho} \right\rfloor \rho = (n \bmod \rho) < \rho$, we have $1 \le j < \rho$. Furthermore, since $i = \rho x_i + r_i = \rho \left\lfloor \frac{n}{\rho} \right\rfloor + j$, the uniqueness of the representation of i as given by (77) implies that

$$x_i = \left\lfloor \frac{n}{\rho} \right\rfloor \quad \text{and} \quad r_i = j$$

Therefore,

$$\begin{aligned}\frac{1}{\rho}\sum_{i=\left\lfloor\frac{n}{\rho}\right\rfloor\rho+1}^{n} r_i &= \frac{1}{\rho}\sum_{j=1}^{n-\left\lfloor\frac{n}{\rho}\right\rfloor\rho} j \\ &= \frac{\left(n-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)\left(n+1-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)}{2\rho} \end{aligned} \tag{83}$$

Substituting (83) into (82), we obtain

$$\frac{1}{\rho}\sum_{i=1}^{n} r_i = \left\lfloor\frac{n}{\rho}\right\rfloor\frac{(\rho+1)}{2} + \frac{\left(n-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)\left(n+1-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)}{2\rho} \tag{84}$$

Note that the second term on the right hand side of (84) vanishes when n/ρ is an integer and (84) in such a case becomes (81). Hence (84) can be used to evaluate $\frac{1}{\rho}\sum_{i=1}^{n} r_i$ for the two discussed cases. Substituting (84) into (80), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left\lceil\frac{i}{\rho}\right\rceil &= n+\frac{n(n+1)}{2\rho}-\left\lfloor\frac{n}{\rho}\right\rfloor\left(\frac{\rho+1}{2}\right)-\frac{1}{2\rho}\left[\left(n-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)\left(n+1-\left\lfloor\frac{n}{\rho}\right\rfloor\rho\right)\right] \\ &= n+\left(n-\frac{\rho}{2}\right)\left\lfloor\frac{n}{\rho}\right\rfloor-\frac{\rho}{2}\left\lfloor\frac{n}{\rho}\right\rfloor^2 \end{aligned} \tag{85}$$

If $X = n \bmod \rho$, then $\left\lfloor\frac{n}{\rho}\right\rfloor = \frac{n-X}{\rho}$. Simplifying Eq. (85), we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left\lceil\frac{i}{\rho}\right\rceil &= n+\frac{n-X}{\rho}\left(n-\frac{\rho}{2}-\frac{n-X}{2}\right) \\ &= n+\frac{n-X}{\rho}\left(\frac{n}{2}-\frac{\rho-X}{2}\right) \\ &= n+\frac{n^2}{2\rho}-\frac{nX}{2\rho}-\frac{n}{2}+\frac{nX}{2\rho}+\frac{X(\rho-X)}{2\rho} \\ &= \frac{n}{2}+\frac{n^2}{2\rho}+\frac{X(\rho-X)}{2\rho} \end{aligned} \tag{86}$$

Proof of Lemma 8:
Consider the following cases of m:

1. m - odd

$$f(m,\rho) = 2\sum_{i=1}^{\frac{m-1}{2}}\left\lceil\frac{i}{\rho}\right\rceil$$

Substituting $n = \frac{m-1}{2}$ in Eq. (17), we get

$$\begin{aligned} f(m,\rho) &= 2\left(\frac{m-1}{4}+\frac{(m-1)^2}{8\rho}+\frac{X(\rho-X)}{2\rho}\right) \quad \text{where } X = \tfrac{m-1}{2} \bmod \rho \\ &= \frac{m-1}{2}+\frac{(m-1)^2}{4\rho}+\frac{X(\rho-X)}{\rho} \end{aligned} \tag{87}$$

Since $X = 0$, we have

$$f(m,\rho) = \frac{m-1}{2} + \frac{(m-1)^2}{4\rho} \quad (88)$$

2. m- even

$$f(m,\rho) = 2\sum_{i=1}^{\frac{m}{2}} \left\lceil \frac{i}{\rho} \right\rceil - \left\lceil \frac{m}{2\rho} \right\rceil$$

Substituting $n = \frac{m}{2}$ in Eq. (17)

$$f(m,\rho) = 2\left(\frac{m}{4} + \frac{m^2}{8\rho} + \frac{X(\rho - X)}{2\rho}\right) - \frac{m}{2\rho} \quad \text{where } X = \tfrac{m}{2} \bmod \rho$$

$$= \frac{m}{2} + \frac{m^2}{4\rho} + \frac{X(\rho - X)}{\rho} - \left\lceil \frac{m}{2\rho} \right\rceil \quad (89)$$

As $X = 0$, Eq. (89) reduces to

$$f(m,\rho) = \frac{m}{2} + \frac{m^2}{4\rho} - \frac{m}{2\rho}$$

$$= \frac{m}{2} + \frac{m(m-2)}{4\rho} \quad (90)$$

■

Appendix C.

Proof of Lemma 9:
From Eq. (89), we know that $X = m/2 \bmod \rho$. Substituing the value of m, we get $X = (\rho(\rho-1)/2) \bmod \rho = ((\rho^2/2) \bmod \rho - (\rho/2) \bmod \rho) \mod \rho = (0 - (\frac{\rho}{2}) \bmod \rho) \bmod \rho = \rho/2$. Hence substituting X and m in Eq. (89), we have

$$f(m,\rho) = \frac{\rho(\rho-1)}{2} + \frac{\rho^2(\rho-1)^2}{4\rho} + \frac{\frac{\rho}{2}(\rho - \frac{\rho}{2})}{\rho} - \left\lceil \frac{\rho(\rho-1)}{2\rho} \right\rceil$$

$$= \frac{\rho(\rho-1)}{2} + \frac{\rho(\rho-1)^2}{4} + \frac{\rho}{4} - \frac{\rho}{2}$$

$$= \frac{\rho(\rho-1)}{2}\left(\frac{\rho+1}{2}\right) - \frac{\rho}{4}$$

$$= \frac{\rho}{4}\left(\rho^2 - 2\right) \quad (91)$$

■

Proof of Lemma 10:
Note that $X = (m+v)/2 \bmod \rho = \rho^2/2 \bmod \rho = 0$ since q is even. Hence Eq. (89) reduces to the following

$$f(m,\rho) = \frac{\rho^2}{2} + \frac{\rho^2(\rho^2-2)}{4\rho}$$

$$= \frac{\rho}{4}\left(\rho^2 + 2\rho - 2\right) \tag{92}$$

■

Proof of Lemma 11:
Observe that $m - \lambda = (\rho - 1)^2$ is odd and hence $X = \lfloor (m-\lambda)/2 \rfloor \bmod \rho$. This can be simplified as $X = (\rho-1)^2 - 1)/2 \bmod \rho = (\rho^2 - 2\rho)/2 \bmod \rho = 0$ since ρ is even. Hence Eq. (88) reduces to

$$\begin{aligned} f(m,\rho) &= \frac{m-\lambda-1}{2} + \frac{(m-\lambda-1)^2}{4\rho} \\ &= \frac{(\rho-1)^2-1}{2} + \frac{(\rho-1)^2-1)^2}{4\rho} \\ &= \frac{\rho(\rho-2)}{2} + \frac{\rho(\rho-2)^2}{4} \\ &= \frac{\rho^2(\rho-2)}{4} \end{aligned} \tag{93}$$

■

Proof of Lemma 12:
Evaluating $X = m/2 \bmod \rho$, we obtain $X = ((\rho^2 - 3\rho + 2)/2) \bmod \rho = (\rho+2)/2$. Substituting the values of X and m in Eq. (89), we get

$$\begin{aligned} f(m,\rho) &= \frac{(\rho-1)(\rho-2)}{2} + \frac{(\rho-1)^2(\rho-2)^2}{4\rho} + \frac{\frac{(\rho+2)}{2}\left(\rho - \frac{(\rho+2)}{2}\right)}{\rho} - \left\lceil \frac{(\rho-1)(\rho-2)}{2\rho} \right\rceil \\ &= \frac{\rho^2-3\rho+2}{2} + \frac{(\rho^2-2\rho+1)(\rho^2-4\rho+1)}{4\rho} + \frac{\rho^2-4}{4\rho} - \frac{(\rho-2)}{2} \\ &= \frac{\rho^3 - 4\rho^2 + 6\rho - 4}{4} \\ &= \frac{(\rho-2)\,(\rho^2-2\rho+2)}{4} \end{aligned} \tag{94}$$

■

Proof of Lemma 13:
Since $m - \lambda = (\rho-2)^2$ is even, we have $X = ((\rho^2 - 4\rho + 4)/2) \bmod \rho$. In other words, $X = (\rho^2/2 - 2\rho + 2) \bmod \rho = 2$. Substituting $X = 2$ and $m - \lambda = (\rho-2)^2$ in Eq. (89), we obtain

$$\begin{aligned} f(m-\lambda,\rho) &= \frac{(\rho-2)^2}{2} + \frac{(\rho-2)^4}{4\rho} + \frac{2(\rho-2)}{\rho} - \left\lceil \frac{(\rho-2)^2}{2\rho} \right\rceil \\ &= \frac{\rho^2-4\rho+4}{2} + \frac{(\rho^2-4\rho+4)^2}{4\rho} + \frac{2(\rho-2)}{\rho} - \frac{(\rho-2)}{2} \\ &= \frac{\rho^3 - 6\rho^2 + 14\rho - 12}{4} \\ &= \frac{(\rho-2)(\rho^2-4\rho+6)}{4} \end{aligned} \tag{95}$$

■

Proof of Lemma 14:
Evaluating $X = m/2 \bmod \rho$, we obtain $X = ((\rho^2+\rho)/2) \bmod \rho = \rho/2$. Substituting X and m in Eq. (89), we get

$$\begin{aligned} f(m,\rho) &= \frac{\rho(\rho+1)}{2} + \frac{\rho^2(\rho+1)^2}{4\rho} + \frac{\frac{\rho}{2}\left(\rho - \frac{\rho}{2}\right)}{\rho} - \left\lceil \frac{\rho(\rho+1)}{2\rho} \right\rceil \\ &= \frac{\rho(\rho+1)}{2} + \frac{\rho(\rho^2+2\rho+1)}{4} + \frac{\rho}{4} - \frac{(\rho+2)}{2} \\ &= \frac{\rho^3+4\rho^2+2\rho-4}{4} \\ &= \frac{(\rho+2)\,(\rho^2+2\rho-2)}{4} \end{aligned} \tag{96}$$

■

Proof of Lemma 15:
Here $X = \lfloor (\rho+1)^2/2 \rfloor \bmod \rho = (((\rho+1)^2-1)/2) \bmod \rho = (\rho^2+2\rho)/2 \bmod \rho = 0$ since q is even. Hence Eq. (88) reduces to

$$\begin{aligned} f(m,\rho) &= \frac{(\rho+1)^2-1}{2} + \frac{((\rho+1)^2-1)^2}{4\rho} \\ &= \frac{\rho^2+2\rho}{2} + \frac{(\rho^2+2\rho)^2}{4\rho} \\ &= \frac{\rho^3+6\rho^2+8\rho}{4} \\ &= \frac{\rho(\rho+2)(\rho+4)}{4} \end{aligned} \tag{97}$$

■

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